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NORMAL COORDINATES ALONG A GEODESIC

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A b s t r a c t

The so called normal coordinates where derivatives of the metric tensor $g_{\mu\nu}$ are expressed through the curvature tensor and its derivatives, are constructed along a geodesic. Explicit expressions for the derivatives of $g_{\mu\nu}$ up to the fourth order are presented and recursive relations permitting a simple calculation of the higher derivatives are found.

As is well known, for a given point $\mathcal{P}$ in a Riemannian space one can always find the coordinates where the Christoffel symbols vanish at the point considered

$$\Gamma^{\mu}_{\alpha\beta}\Big|_{\mathcal{P}} = 0 \tag{1}$$

This is a mathematical expression of the equivalence principle in General Relativity. The coordinates satisfying condition (1) are called geodesic coordinates.

For a given point there exist infinitely many different geodesic coordinate systems. Using this freedom one can make a further specialization of the coordinates. In particular Riemann normal coordinates at point $\mathcal{P}$ can be constructed. First derivatives of the metric tensor at point $\mathcal{P}$ vanish in these coordinates and the higher derivatives are expressed through the curvature tensor $R_{\alpha\beta\gamma\delta}$ and its derivatives. The following expansion of the metric in powers of small deviation from point $\mathcal{P}$ is known

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^{\alpha} x^{\beta} - \frac{1}{3!} R_{\mu\alpha\nu\beta,\gamma}\, x^{\alpha} x^{\beta} x^{\gamma} + \frac{1}{5!}\Big\{ -6 R_{\mu\alpha\nu\beta,\gamma\delta} + \frac{4}{3} R_{\mu\alpha\sigma\beta} R^{\sigma}{}_{\gamma\nu\delta} \Big\} x^{\alpha} x^{\beta} x^{\gamma} x^{\delta} + \ldots \tag{2}$$

where $\eta_{\mu\nu}$ is a metric of flat space*), and $R_{\mu\alpha\nu\beta,\gamma} = \partial R_{\mu\alpha\nu\beta}/\partial x^{\gamma}$ at $x^{\sigma} = 0$ and so on.

The quadratic in x^{α} term in this expression was found by Eddington (1923) and the cubic and quartic ones were calculated by Petrov (1961, 1969). Here the numerical coefficient at the nonlinear in the curvature tensor term differs from that presented in the book by Petrov (1961, 1969) because in this book the result is given in terms of covariant derivatives and

*) The sign convention is the following: $ds^2 > 0$ for a timelike interval and $R_{\alpha\beta\gamma\delta}$ is defined according to Eq. (10).

in our paper usual derivatives are used.

As it was shown by Fermi (1922) (see also Rashevsky (1967)), it is possible to introduce such coordinate systems that Christoffel symbols vanish not only at a single point, but along an arbitrary curve γ

$$\Gamma^{\mu}_{\alpha\beta}\Big|_{\gamma} = 0 \tag{3}$$

In what follows we consider the case where γ is a geodesic, and in a number of the coördinate systems which satisfy condition (3), find so called normal coordinates around geodesic γ. In full analogy with normal coordinates at a point these coordinates permit to express derivatives of the metric tensor through the curvature tensor and its derivatives, however, not only at one point, but along the whole curve γ. Such coordinate systems were first introduced, as far as we know, by Manasse and Misner (1963) who call them Fermi normal coordinates. In what follows we further develope this notion.

Our paper differs from that by Manasse and Misner (1963) in the following: a) we have explicitly presented the expression of $g_{\mu\nu}$ in powers of distance normal to the geodesic up to the quartic terms whereas Manasse and Misner (1963) have calculated only second order terms; moreover, we have calculated the arbitrary coefficient of the expansion in the case of a weak gravitational field (that is taking into account only terms linear in the curvature tensor); a recursive relation has been derived which permits one to calculate the expansion coefficients order by order in any gravitational field; b) we present here two different ways of derivation of expansion of type (2) around any geodesic γ; the first of these ways is based on the explicit construction of the corresponding coordinate system and the second, more formal one, is analogous to the treating of gauge fields in the Schwinger gauge (Dubovikov and Smilga (1981)); c) in that paper a world line of a massive test particle was taken as the basic geodesic γ, i.e. the latter was described by the equations $x=y=z=0$ and t is arbitrary; in our paper the type of the geodesic is not specified, in particular it can be a light geodesic which is con-

venient for the analysis of electromagnetic wave propagation in a gravitational field; this is not essential however because the relation between the metric and the curvature tensors does not depend on the form of γ.

First we construct a coordinate system where Christoffel symbols vanish on a given geodesic γ (see Eq. (3)). The consideration is essentially the same as in book [4]. We however reproduce it because some of the points can be interesting for what follows.

Let geodesic γ be defined by the equation*)

$$x^p = 0 \quad (p = 0,1,2) \tag{4}$$

that is γ coincides with the coordinate axis x^3. In this coordinate system γ is a geodesic if

$$\Gamma^{\alpha}_{33} = 0 \tag{5}$$

Introduce new coordinates according to the relation:

$$x'^{\mu} = \delta^{\mu}_{3} x^3 + a^{\mu}_{p}(x^3)x^p + \frac{1}{2} b^{\mu}_{pq}(x^3)x^p x^q \tag{6}$$

Evidently in terms of these coordinates geodesic γ is defined by the formally same equation $x'^p = 0$.

It follows from the transformation law

$$\Gamma'^{\mu}_{\sigma\lambda} = \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} \left(\Gamma^{\alpha}_{\beta\gamma} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} - \frac{\partial^2 x'^{\mu}}{\partial x^{\beta} \partial x^{\gamma}} \right) \tag{7}$$

that $\Gamma'^{\mu}_{33} = 0$ if Eq. (5) is valid. Moreover all the components $\Gamma'^{\alpha}_{\beta\gamma}$ vanish on γ in coordinates (6) if functions $a^{\nu}_{p}(x^3)$ and $b^{\nu}_{pq}(x^3)$ satisfy the equations

$$\frac{\partial a^{\nu}_{p}}{\partial x^3} = \Gamma^{q}_{3p} a^{\nu}_{q} + \Gamma^{3}_{3p} \delta^{\nu}_{3}$$

*) Here and what follows Latin indices denote components normal to geodesic (p, q, r, etc. = 0,1,2) while Greek indices give all the four coordinates (α, β, γ, etc. = 0, 1, 2, 3).

$$b^{\nu}_{pq} = \Gamma^{3}_{pq}\delta^{\nu}_{3} + a^{\nu}_{s}\Gamma^{s}_{pq}$$

Consequently the metric on γ is the metric of flat space-time: $g_{\mu\nu} = \eta_{\mu\nu}$.

Note that if γ is not a geodesic, one can nevertheless find coordinates where condition (3) is valid, but in these coordinates the equations which determine γ can not be in general written as $x^{p} = 0$.

In what follows we suppress primes by which new coordinates (6) are denoted, saving primes for the second generation in which the second derivatives of $g_{\mu\nu}$ on γ can be expressed through the curvature tensor. This procedure of shifting the primes will be repeated as many times as necessary in introducing newer and newer coordinate systems.

The set $\partial_{\alpha}\partial_{\beta}g_{\mu\nu}$ consists of a hundred of functions whereas the number of independent components of the curvature tensor is equal to 20. So generally speaking $\partial_{\alpha}\partial_{\beta}g_{\mu\nu}$ cannot be expressed through $R_{\alpha\mu\beta\nu}$. The coordinate freedom however permits to exclude extra components of $\partial_{\alpha}\partial_{\beta}g_{\mu\nu}$ and to get the connection we are interested in. Because of the symmetry properties of $\partial_{\alpha}\partial_{\beta}g_{\mu\nu}$ and $R_{\alpha\mu\beta\nu}$ this connection should be of the form

$$\partial_{\alpha}\partial_{\beta}g_{\mu\nu} \equiv g_{\mu\nu,\alpha\beta} = K\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\right) \qquad (8)$$

where K is a numerical factor which depends upon the values of μ and ν (see below).

Evidently the following equations hold on γ

$$\partial_{3}\partial_{3}g_{\mu\nu} = \partial_{3}\partial_{p}g_{\mu\nu} = 0$$

So in Eq. (8) we consider only $\alpha, \beta = 0,1,2$.

One can easily see that in the chosen above coordinate system the relation holds

$$g_{33,pq} = 2R_{3pq3} \qquad (9)$$

which follows from the general expression

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}\right) + \qquad (10)$$
$$+ g_{\mu\nu}\left(\Gamma^{\mu}_{\alpha\delta}\Gamma^{\nu}_{\beta\gamma} - \Gamma^{\mu}_{\alpha\gamma}\Gamma^{\nu}_{\beta\delta}\right)$$

To derive Eq.(8) when μ or ν (or both) are not equal to 3, a further modification of the coordinate system is needed. It can be achieved through the transformation

$$x'^{\mu} = x^{\mu} + \frac{1}{6}a^{\mu}_{pqr}(x^{3})x^{p}x^{q}x^{r} \qquad (11)$$

Note that the new coordinates differ from the old ones only in cubic in x^{p} terms and because of this Eqs. (3) and (9) keep to be true after transformation (11).

Up to quadratic in x^{p} terms the metric tensor transforms as

$$g_{\mu\nu} = g'_{\mu\nu} + \frac{1}{2}x^{p}x^{q}\left(g'_{\alpha\nu}\delta^{r}_{\mu}a^{\alpha}_{pqr} + g'_{\alpha\mu}\delta^{r}_{\nu}a^{\alpha}_{pqr}\right) \qquad (12)$$

To see what restrictions have to be imposed on a^{α}_{pqr} , let us substitute Eq. (10) into (8). It is convenient to discuss the cases $\mu = 3$, $\nu \neq 3$ and $\mu \neq 3$, $\nu \neq 3$ separately. In the first case relation (8) is valid if

$$g'_{3r,pq} + g'_{3p,qr} + g'_{3q,rp} = 0 \qquad (13)$$

and the latter is satisfied for $a_{3pqr} = g_{\alpha 3}a^{\alpha}_{pqr}$ chosen as

$$a_{3pqr} = \frac{1}{3}\left(g_{3r,pq} + g_{3p,qr} + g_{3q,pr}\right) \qquad (14)$$

Note that under change of coordinates (11) the equality holds

$$\frac{\partial^{2}g'_{\mu\nu}}{\partial x^{p}\partial x^{q}} = \frac{\partial^{2}g'_{\mu\nu}}{\partial x'^{p}\partial x'^{q}}$$

For $\mu \neq 3$ and $\nu \neq 3$ the derivatives of the metric tensor in the new coordinate system are to satisfy the conditions

$$g'_{pq,rs} = g'_{rs,pq} \tag{15a}$$

$$g'_{pq,rs} + g'_{pr,sq} + g'_{ps,qr} = 0 \tag{15b}$$

which in turn are provided by

$$a_{spqr} = g_{s\sigma} a^{\sigma}_{pqr} = \frac{1}{3}\left(g_{sp,qr} + g_{sq,pr} + g_{sr,pq}\right) - \frac{1}{6}\left(g_{pq,sr} + g_{rp,sq} + g_{rq,sp}\right) \tag{16}$$

Thus substituting Eq. (10) into Eq. (8) and using Eqs. (13) and (15), we obtain

$$g_{3r,pq} = -\frac{2}{3}\left(R_{3pqr} + R_{3qrp}\right) \tag{17a}$$

$$g_{sr,pq} = -\frac{1}{3}\left(R_{sprq} + R_{sqrp}\right) \tag{17b}$$

along the geodesic γ .

These equations together with Eq. (9) solve the problem of expressing $g_{\mu\nu}$ through the curvature tensor up to quadratic terms in deviation from the geodesic γ . The result obtained agrees with that of Manasse and Misner (1963).

Based on this example let us calculate higher derivatives of $g_{\mu\nu}$. First we introduce the following notation. If k indices of a tensor are denoted by stars, then total symmetrization in these indices is performed, i.e. summation in all permutations of the indices and division by $k!$, the symmetrization being made only for Latin indices, i.e. having values 0,1,2. For example

$$T_{\alpha*\beta**} = \frac{1}{3!}\left(T_{\alpha p\beta qr} + T_{\alpha p\beta rq} + T_{\alpha q\beta pr} + T_{\alpha q\beta rp} + T_{\alpha r\beta pq} + T_{\alpha r\beta qp}\right) \tag{18}$$

According to such a definition

$$g_{\alpha\beta,p_1p_2\dots p_k} = g_{\alpha\beta,\underbrace{**\dots*}_{k}} \tag{19}$$

Consider a $(k-2)$-th derivative of the curvature tensor. Due to Eq. (10) we have

$$R_{\alpha*\beta*,*\dots*} = \frac{1}{2}\left(g_{\alpha*,\beta*\dots*} + g_{\beta*,\alpha*\dots*} - g_{\alpha\beta,*\dots*} - g_{**,\alpha\beta*\dots*}\right) - \frac{1}{4}\Big[g^{\mu\sigma}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right)\left(g_{\sigma*,*} + g_{\sigma*,*} - g_{**,\sigma}\right) - g^{\mu\sigma}\left(g_{\mu\alpha,*} + g_{\mu*,\alpha} - g_{\alpha*,\mu}\right)\left(g_{\sigma\beta,*} + g_{\sigma*,\beta} - g_{\beta*,\sigma}\right)\Big]_{,**\dots*} \tag{20}$$

Now let us demand for any $k \geq 2$ the following conditions to be fulfilled along γ

$$g_{\alpha*,\underbrace{*\dots*}_{k-1}} = 0 \tag{21}$$

$$g_{**,\alpha\underbrace{*\dots*}_{k-2}} = 0 \tag{22}$$

The first of them is the generalization of Eqs. (13) and (15b). Note that for $\alpha = 3$ condition (22) is not independent, but can be obtained from Eq. (21) by differentiating it in x^3.

From these equations another two can be obtained

$$g_{3*,p\underbrace{*\dots*}_{k-1}} = -\frac{1}{k}\, g_{3p,\underbrace{*\dots*}_{k}} \tag{23}$$

$$g_{**,pq\underbrace{*\dots*}_{k-2}} = \frac{2}{k(k-1)}\, g_{pq,\underbrace{*\dots*}_{k}} \tag{24}$$

the second of which is the generalization of Eq. (15a).

Below we check the selfconsistency of these conditions, that is we show that the expressions obtained for $g_{\alpha\beta,p_1\dots p_k}$ with the help of relations (21)-(24) in fact satisfy them. We also explicitly construct the coordinate system where Eqs. (21)-(24) take place.

It is evident that Eqs (21)-(24) can be differentiated in x^3 (but not in x^p). Keeping this in mind, we obtain

$$g_{33,*\dots*} = -2R_{3*3*,*\dots*} + \frac{1}{2}\Big[g^{\mu\sigma}(g_{\mu 3,*} - g_{3*,\mu})(g_{\sigma 3,*} - g_{3*,\sigma})\Big]_{,*\dots*} \tag{25}$$

$$g_{3p,*\dots*} = \frac{k}{k+1}\Big\{-2R_{3*p*,*\dots*} + \frac{1}{2}\Big[g^{\mu\sigma}(g_{\mu 3,*} - g_{3*,\mu})(g_{\sigma p,*} + g_{\sigma *,p} - g_{p*,\sigma})\Big]_{,*\dots*} \tag{26}$$

and at last

$$g_{pq,*\dots*} = \frac{k-1}{k+1}\Big\{-2R_{p*q*,*\dots*} + \frac{1}{2}\Big[g^{\mu\sigma}(g_{\mu p,*} + g_{\mu *,p} - g_{p*,\mu})(g_{\sigma q,*} + g_{\sigma *,q} - g_{p*,\sigma})\Big]_{,*\dots*}\Big\} \tag{27}$$

Expressions (25)-(27) solve our problem because they determine k-th derivatives of $g_{\mu\nu}$ through the curvature tensor and lower derivatives of $g_{\mu\nu}$. The latter give rise to terms of the second and higher orders in powers of $R_{\alpha\beta\gamma\delta}$ and are non-

vanishing only for $k \geq 4$.

Now the following expansion of interval ds^2 in powers of small deviations x^p from the geodesic γ can be written

$$\begin{aligned} ds^2 = {} & (dx^3)^2\Big[\eta_{33} + \frac{2}{2!}R_{3pq3}\,x^p x^q + \frac{2}{3!}R_{3pq3,r}\,x^p x^q x^r + \\ & + \frac{2}{4!}\big(R_{3pq3,rs} + 2\eta^{\mu\sigma}R_{3p\mu q}R_{3r\sigma s}\big)x^p x^q x^r x^s\Big] + \\ & + 2dx^3 dx^j\Big[\eta_{3j} + \frac{4}{3}\frac{1}{2!}R_{3pqj}\,x^p x^q + \frac{3}{2}\frac{1}{3!}R_{3pqj,r}\,x^p x^q x^r + \\ & + \frac{8}{5}\frac{1}{4!}\big(R_{3pqj,rs} + \frac{2}{3}\eta^{\mu\sigma}R_{\mu p3q}R_{\sigma rj}\big)x^p x^q x^r x^s + \\ & + dx^i dx^j\Big[\eta_{ij} + \frac{2}{3}\frac{1}{2!}R_{ipqj}\,x^p x^q + \frac{1}{3!}R_{ipqj,r}\,x^p x^q x^r + \\ & + \frac{6}{5}\frac{1}{4!}\big(R_{ipqj,rs} + \frac{2}{9}\eta^{\mu\sigma}R_{\mu piq}R_{\sigma rjs}\big)x^p x^q x^r x^s\Big] + \dots \end{aligned} \tag{28}$$

Note that the coefficients of the expansion of g_{ij} in powers of x^p near the geodesic γ must coincide with these of the expansion of $g_{\mu\nu}$ in powers of x^α in Riemann normal coordinates at a given point. Our result for g_{ij} is indeed the same as the one presented in the book by Petrov (1961, 1969).

Now we see that due to the known symmetry properties of the curvature tensor, expressions (26) and (27) satisfy conditions (21) and (22). The latter to be fulfilled we have to choose a proper coordinate system. For $k = 3$ the transition from the coordinates where Eqs. (9) and (17) already take place, to coordinates where in addition Eqs. (25)-(27) are valid, can be done by the transformation

$$x'^\mu = x^\mu + \frac{1}{4!}a^\mu{}_{pqrs}\,x^p x^q x^r x^s \tag{29}$$

with

$$a_{\mu****} \equiv g_{\mu\sigma} a^{\sigma}_{****} = g_{\mu*,***} - \frac{1}{2}\delta^{t}_{\mu} g_{**,t**}$$

Analogous expressions can be written for higher values of k.

Expansion (2) (as well as (28)) can be also obtained in the pure algebraic way. Let the coordinate system be fixed by the condition

$$(g_{\mu\nu} - \eta_{\mu\nu})x^{\mu} = 0 \tag{30}$$

Note that it resembles the condition which defines the Schwinger gauge in gauge field theory, $x^{\mu}A_{\mu} = 0$. Of course expression (2) satisfies condition (30).

From Eq. (30) we find by differentiation and multiplication by x^{λ} that

$$g_{\mu\nu,\lambda_1\ldots\lambda_k} x^{\nu} x^{\lambda_1}\ldots x^{\lambda_k} = 0 \tag{31}$$

$$g_{\mu\nu,\sigma\lambda_1\ldots\lambda_k} x^{\mu} x^{\nu} x^{\lambda_1}\ldots x^{\lambda_k} = 0 \tag{32}$$

in analogy with conditions (21) and (22).

Using these equations, it is easy to obtain

$$x^{\mu} x^{\nu} g_{\mu\nu,\alpha\beta} = 2(g_{\alpha\beta} - \eta_{\alpha\beta}) \tag{33}$$

Reexpressing $g_{\mu\nu,\alpha\beta}$ through the curvature tensor, we find

$$2g_{\alpha\beta,\mu} x^{\mu} + g_{\alpha\beta,\mu\nu} x^{\mu} x^{\nu} = 2R_{\mu\alpha\beta\nu} x^{\mu} x^{\nu} + \frac{1}{2} x^{\mu} x^{\nu} g^{\sigma\lambda} g_{\sigma\alpha,\mu} g_{\lambda\beta,\nu} \tag{34}$$

The expansion of both sides of this equation in powers of x gives just recursive relation (27) which in turn leads to expression (2).

Equation (34) looks somewhat more simple in terms of functions $f_{\alpha\beta} = x^{\mu} g_{\alpha\beta,\mu}$. Following the procedure of Dubovikov and Smilga (1981), we make the change of variables $x^{\lambda} \to a x^{\lambda}$ and find that the l.h.s. of Eq. (34) is equal to $\frac{\partial}{\partial a}\left[a f_{\alpha\beta}(ax)\right]$ at $a = 1$. Thus the integral relation between $f_{\alpha\beta}$ and $R_{\mu\alpha\beta\nu}$ can be obtained

$$f_{\alpha\beta}(x) = \int_0^1 da \Big\{ 2R_{\mu\alpha\beta\nu}(ax) a^2 x^{\mu} x^{\nu} + \frac{1}{2} a^2 x^{\mu} x^{\nu} g^{\sigma\lambda}(ax) f_{\alpha\sigma}(ax) f_{\beta\lambda}(ax) \Big\} \tag{34}$$

In contrast to vector gauge fields for which the potential A_{μ} if the condition $A_{\mu}x^{\mu} = 0$ is valid, can be expressed through the integral from the field strength as

$$A_{\mu}(x) = \int_0^1 da\, a x^{\nu} G_{\nu\mu}(ax), \tag{35}$$

the nonlinear term survives in Eq. (34). This equation however can be used for perturbative calculation of $f_{\alpha\beta}$:

$$\begin{aligned} f_{\alpha\beta}(x) = x^{\mu} x^{\nu} \int_0^1 da\, a^2 \Big\{ & 2R_{\mu\alpha\beta\nu}(ax) + \\ & + 2\eta^{\sigma\lambda} \int_0^1 db_1\, b_1^2 x^{\sigma_1} x^{\sigma_2} R_{\sigma_1\sigma\alpha\sigma_2}(ab_1 x) \cdot \\ & \int_0^1 db_2\, b_2^2 x^{\lambda_1} x^{\lambda_2} R_{\lambda_1\lambda\beta\lambda_2}(ab_2 x) + \ldots \Big\} \end{aligned} \tag{36}$$

It is noteworthy that in the coordinate system fixed by condition (30) the metric tensor can be expressed in terms of the Christoffel symbols as

$$g_{\alpha\beta} - \eta_{\alpha\beta} = -x^{\mu}(\Gamma_{\mu,\alpha\beta} + \Gamma_{\beta,\alpha\mu}) \tag{37}$$

The symmetry of $g_{\alpha\beta}$ is guaranteed by the condition

$$x^{\mu}(\Gamma_{\alpha\beta\mu} - \Gamma_{\beta,\alpha\mu}) = 0$$

Analogous consideration can be used also for derivation of expansion (28) around a geodesic.

The coordinates considered prove to be very convenient for the analysis of electromagnetic wave propagation in a gravitational field when the wave length is small as compared with the characteristic scale of the field. One can see that there are corrections to the phase and group velocity of the electromagnetic wave packet of the order of λ^2 (λ is the electromagnetic wave length). To express these corrections through the curvature tensor one needs to expand $g_{\mu\nu}(x)$ in the Maxwell equations in a gravitational background up to the terms of the fourth order in x^λ. The Maxwell equations in these coordinates can be diagonalized so that one obtains a single equation for a single unknown function. In fact the necessity of finding corrections to the group and phase velocity was the starting point of this investigation.

After the present work was over, our attention was attracted to two additional references [7,8] where normal coordinates are considered. Paper [7] is especially interesting because the authors use the same conditions as our Eq. (30) as a basic point of their construction. The method of Ref. [7] however essentially relies on linear approximation in the curvature tensor. Their result is equivalent to our Eq. (35) with the quadratic term in $f_{\alpha\beta}$ being neglected. Thanks to paper [7] we found one more relevant reference [9] where the conditions coinciding with our Eqs. (15) and (21) were explicitly written down, no expansion of $g_{\mu\nu}$ being presented however. Then Prof. L.Halpern kindly informed us that long ago he obtained the expansion of $g_{\mu\nu}$ up to the third order in x^λ, i.e. to the highest order where nonlinear effects are absent.

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