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QUASICLASSICAL APPROACH TO THE HIGH-ENERGY DELBRÜCK SCATTERING

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A b s t r a c t

A quasiclassical approach intended for description of the quantum electrodynamics processes in the Coulomb field at high energies is developed. An expression for the high-energy Delbruck scattering amplitude is derived in a simple form, thereby allowing to define the dependence of the total cross section on the charge of the Coulomb centre.

The elastic scattering of a photon in the Coulomb field via virtual electron-positron pairs (Delbrück scattering, Ref. /1/) is one among a few nonlinear processes in quantum electrodynamics directly observable in the experiments (see Refs. /2/ and /3/). The situation when $\omega \gg m$ (ω is the photon frequency, m is the mass of an electron; we set $\hbar = c = 1$) is most favourable.

A variety of attempts to theoretically describe this process are surveyed in Ref. /2/, but only Cheng and Wu (see Refs. /4-6/) have solved this problem summing in a definite approximation the Feynman diagrams with arbitrary number of the photon exchange with the Coulomb centre. It appears (Refs. /4/ and /5/) that the Coulomb corrections at $Z\alpha \sim 1$ ($Z|e|$ is the charge of the nucleus, $\alpha = e^2 = 1/137$ is the fine structure constant, e is an electron charge) drastically change the result as compared to the Born approximation.

In the present paper, the Delbrück scattering amplitude at $\omega \gg m$ is found by the method substantially different from that developed by Cheng and Wu. The approach is based on the use of the integral representation for the electron Green's function $G(\vec{r}_2, \vec{r}_1 | \varepsilon)$ in the Coulomb field obtained by the authors in Ref. /7/ and also on taking into account, explicitly, the quasiclassical nature of the motion of high-energy charged particles. The suggested method may be applied to the solution of the other problems in the Coulomb field. In the case under consideration, this approach leads to a substantially simpler expression for the amplitude of the process, that makes it possible to calculate its total cross section.

Let the initial photon with momentum $\vec{K}_1 = \omega \vec{v}_1$ produce a pair of virtual particles at the point $\vec{r}_1$. This pair converts to the photon with momentum $\vec{K}_2 = \omega \vec{v}_2$ ($|\vec{K}_1| = |\vec{K}_2| = \omega$) at the point $\vec{r}_2$. The main contribution to the amplitude comes from the energy of the charged particles $\varepsilon_1 \sim \varepsilon_2 \sim \omega$. Let $\vec{\Delta} = \vec{K}_2 - \vec{K}_1$. Then, one obtains, from the uncertainty relation, that the lifetime of a virtual pair (i.e. the loop length) is $\tau \sim \omega (m^2 + \Delta^2)^{-1}$ and the characteristic impact parameter $\rho \sim 1/\Delta$. It follows from this that within the domain of momentum transfers

$$m^2/\omega \ll \Delta \ll \omega \tag{1}$$

the ratio $\rho/r \ll 1$, i.e. the angles between vectors $\vec{k}_1, \vec{k}_2, \vec{r}_2$ and $(-\vec{r}_1)$ are small. The relation (1) determines all the physically interesting domain of momentum transfers. In particular, the contribution to the total cross section of momentum transfers not satisfying the condition (1) is suppressed as m^2/ω^2 . It should bear in mind that upon scattering on atoms the point-charge approximation holds if $r_c^{-1} \ll \Delta \ll R^{-1}$ where R is the radius of the nucleus, r_c is the nucleus screening radius (in the Thomas-Fermi model $r_c \sim (m\alpha)^{-1} Z^{-1/3}$. The latter restriction ensures, for $\omega \gtrsim$ 100 MeV, the satisfiability of condition (1), and in what follows we shall therefore consider only this domain of momentum transfers. The domain of momentum transfers $\Delta \lesssim m^2/\omega$ has been examined in Ref. /6/.

In the Furry representation the amplitude of Delbrück scattering is

$$M = 2i\alpha \int d^3r_1 d^3r_2\, e^{i(\vec{k}_1\vec{r}_1 - \vec{k}_2\vec{r}_2)} \int d\varepsilon_1 d\varepsilon_2\, \delta(\omega-\varepsilon_1+\varepsilon_2)\cdot \tag{2}$$
$$\cdot \mathrm{Tr}\left[\hat{e}_1^* G(\vec{r}_1,\vec{r}_2|\varepsilon_2)\hat{e}_2 G(\vec{r}_2,\vec{r}_1|\varepsilon_1)\right]$$

here $\vec{e}_{1,2}$ are the vectors of photon polarization. The integral over $\varepsilon_{1,2}$ goes under the real axis in the left half-plane and above it in the right one. Let us represent the δ - function in eq. (2) as follows:

$$\delta(\omega-\varepsilon_1+\varepsilon_2) = \frac{i}{2\pi}\left[\frac{1}{\omega-\varepsilon_1+\varepsilon_2+i0} - \frac{1}{\omega-\varepsilon_1+\varepsilon_2-i0}\right] \tag{3}$$

Using the first term in eq. (3) and taking into account the analytical properties of the function $G(\vec{r}_1,\vec{r}_2|\varepsilon)$ (see e.g. /7/), the contour of integration over $\varepsilon_1, \varepsilon_2$ in eq. (2) can be deformed so that it will go around the right and the left cuts over ε_1 and ε_2 respectively. Moreover, in integrating over ε_1 there appears the contribution of the discrete spectrum, which can be neglected at $\omega \gg m$. With the second term in eq. (3), the contour of integration over $\varepsilon_1, \varepsilon_2$ in eq. (2) goes after deformation around the left (over ε_1) and the right (over ε_2) cuts. The quantity $\omega-\varepsilon_1+\varepsilon_2$ turns out to be large and the contribution of this term can be neglected. After this

transformations and the substitution $\varepsilon_2 \to -\varepsilon_2$, we have

$$M = \frac{\alpha}{\pi}\int d^3r_1 d^3r_2\, e^{i(\vec{k}_1\vec{r}_1 - \vec{k}_2\vec{r}_2)} \int_m^\infty \frac{d\varepsilon_1 d\varepsilon_2}{\omega-\varepsilon_1-\varepsilon_2+i0} \mathrm{Tr}\left[\hat{e}_1 \delta G(\vec{r}_1,\vec{r}_2|-\varepsilon_2)\hat{e}_2 \delta G(\vec{r}_2,\vec{r}_1|\varepsilon_1)\right] \tag{4}$$

Here δG is the discontinuity of the Green's function on the cut: $\delta G(\varepsilon) = G^{(+)}(\varepsilon) - G^{(-)}(\varepsilon)$, where $G^{(\pm)}(\varepsilon)$ determines the electron Green's function in a Coulomb field, correspondingly, in the upper and lower half-plane of the complex variable ε . Expression (4) corresponds to the diagram of non-covariant perturbation theory, which gives the main contribution at $\omega \gg m$. It's convenient to perform the calculations in terms of helical amplitudes. Let us choose the polarization vectors in the form $\vec{e}^{(\pm)}_{1,2} = \frac{1}{\sqrt{2}}(\vec{\lambda}\times\vec{v}_{1,2} \pm i\vec{\lambda})$, where $\vec{\lambda} = \vec{v}_1\times\vec{v}_2/|\vec{v}_1\times\vec{v}_2|$. There are two independent amplitudes, namely, $M_1 = M_{++} = M_{--}$ and $M_2 = M_{+-} = M_{-+}$. In terms of linear polarizations, by virtue of parity conservation, the amplitude differs from zero only if the polarizations of the initial and final photons lie both in the scattering plane ($M_\parallel$) or are perpendicular to it ($M_\perp$). The following relations hold: $M_1 = (M_\parallel + M_\perp)/2$, $M_2 = (M_\parallel - M_\perp)/2$. By virtue of the momentum conservation $M(Z=0)=0$ for the case under study $\Delta \neq 0$. It is convenient to subtract, from the integrand for M in formula (4), the value of this integrand at Z=0. It is this difference for which the above statement on the smallness of the angles (between vectors $\vec{k}_1, \vec{k}_2, \vec{r}_2$ and $(-\vec{r}_1)$), giving the main contribution to the integral, is valid. In the following such a subtraction is assumed to be made and we take it into account in the explicit form in the final result.

It is worth noting that, according to eq. (1), the characteristic value of the angular momentum $\ell \sim \rho\omega \sim \omega/\Delta$ proves to be large and it is possible to employ the quasiclassical approximation. This means that large values of ℓ contribute to the sum over ℓ in eq. (19) (see Ref. /7/ for $G(\vec{r}_2,\vec{r}_1|\varepsilon)$ and one can neglect $(Z\alpha)^2$ in the quantity $\nu = \sqrt{\ell^2-(Z\alpha)^2}$. After the substitution $\nu \to \ell$ one takes the sum over ℓ by means of formula (24) in Ref. /7/ and we get an expression for the quasiclassical electron Green's function in the Coulomb field:

$$G^{(\pm)}(\vec{r}_2,\vec{r}_1|\varepsilon) = \pm \frac{i\varkappa^2}{4\pi}\int\limits_0^\infty \frac{dt}{\mathrm{sh}^2\varkappa t}\exp\{\pm i[2Z\alpha\varepsilon t + \varkappa(r_2+r_1)\mathrm{cth}\,\varkappa t]\}\cdot$$

$$\cdot\Big\{[\gamma^0\varepsilon + m \pm \frac{\varkappa}{2}(\vec{\gamma},\vec{n}_1-\vec{n}_2)\mathrm{cth}\,\varkappa t]J_0(y) + \frac{iJ_1(y)}{y}\Big[\Big(\frac{\varkappa^2(r_2-r_1)}{2\,\mathrm{sh}^2\varkappa t} + \qquad (5)$$

$$+ Z\alpha m\gamma^0\Big)(\vec{\gamma},\vec{n}_1+\vec{n}_2) \mp Z\alpha\varkappa\,\mathrm{cth}\,\varkappa t\,\gamma^0(1-(\vec{\gamma}\vec{n}_2)(\vec{\gamma}\vec{n}_1))\Big]\Big\}$$

where $\varkappa = \sqrt{\varepsilon^2-m^2}$, $y = \frac{\varkappa}{\mathrm{sh}\,\varkappa t}\sqrt{2r_1r_2(1+\vec{n}_1\vec{n}_2)}$, $\vec{n}_{1,2} = \vec{r}_{1,2}/r_{1,2}$, J_0 and J_1 are Bessel functions, γ^μ are Dirac matrices. The quantities $\varkappa_{1,2}$ in the expression (2) are real and vary from 0 to ∞. The discontinuity $\delta G(\varepsilon)$ is determined by expression (5) if one takes the upper sign in it and the integration over t is carried out from $-\infty$ to ∞. It is appropriate to come to the variable $\varkappa t$ in the integral over t, to make the substitution $\vec{r}_1 \to -\vec{r}_1$ in eq. (4), and to proceed from the integration over $\varepsilon_{1,2}$ to the integration over $\varkappa_{1,2}$. We change over to the variables $R = (r_1r_2)^{1/2}$, $v = (r_1/r_2)^{1/2}$, $P_{1,2} = R\varkappa_{1,2}$. Then, the integral over R takes, within the necessary accuracy, the form

$$\int\limits_0^\infty \frac{dR\,\exp[-i\omega R(v\vec{v}_1\vec{n}_1 + \vec{v}_2\vec{n}_2/v)]}{\omega R - (P_1+P_2)\Big[1 + \frac{m^2}{2\omega^2}\frac{(P_1+P_2)^2}{P_1P_2}\Big] + i0}$$

Here the integration over R can be extended from $-\infty$ to ∞ and the integral is therefore trivial. Then, one takes the trace and expand the integrand in formula (4) in terms of small angles. It is convenient to direct the axis of the spherical coordinate system along $\vec{v}_1+\vec{v}_2$. One has in the small-angles approximation: $d\Omega_{1,2} \simeq \theta_{1,2}\,d\theta_{1,2}\,d\varphi_{1,2} = d^2\vec{\theta}_{1,2}$, $(\vec{\theta}_{1,2},\vec{v}_1+\vec{v}_2) = 0$. The Bessel functions depend on $\vec{\theta}_{1,2}$ in the combination $|\vec{\theta}_1-\vec{\theta}_2|$ only. Let us make the substitution of the variables $\vec{\theta} = \vec{\theta}_2-\vec{\theta}_1$, $\vec{\xi} = v\vec{\theta}_1 + \vec{\theta}_2/v$. After that it is easy to take the integral over $d^2\vec{\xi}$. An analysis of the expression (4) shows that the main contribution to the integral is given by large positive $t_{1,2}$. Let us introduce the variables $x_1 = \frac{1}{2}e^{t_1}$, $x_2 = \frac{1}{2}e^{t_2}$ and perform the expansion, bearing in mind the fact that $x_{1,2} \gg 1$. It is convenient to perform the integration by parts over x_1, x_2 in the expression for M_2 so that only the Bessel function J_1

remains in the preexponent. One obtains

$$M_{1,2} = -\frac{\alpha}{\pi\omega}\int\limits_0^\infty \frac{dx_1}{x_1^3}\frac{dx_2}{x_2^3}\Big(\frac{x_1}{x_2}\Big)^{2iZ\alpha}\int\limits_0^\infty \frac{dP_1\,dP_2\,(P_1P_2)^2}{P_1+P_2}\int\limits_0^\infty \frac{dv}{1+v^2}\int d^2\vec{\theta}\,e^{i\Phi}\,T_{1,2}$$

$$\Phi = \frac{1}{2}\Big(v+\frac{1}{v}\Big)\Big[\frac{P_1}{x_1^2} + \frac{P_2}{x_2^2} + \frac{(P_1+P_2)}{(v+\frac{1}{v})^2}\Big(\vec{\theta}-\frac{\vec{\Delta}}{\omega}\Big)^2 - \frac{m^2}{\omega^2}\frac{(P_1+P_2)^3}{P_1P_2}\Big],$$

$$T_1 = J_0(y_1)J_0(y_2)\Big[\frac{m^2}{\omega^2}\frac{(P_1+P_2)^4}{(P_1P_2)^2} + \frac{1}{2}\Big(\theta^2-\frac{\Delta^2}{\omega^2}\Big) - \frac{1}{x_1^2} - \frac{1}{x_2^2} + \frac{4iv}{(1+v^2)(P_1+P_2)} +$$

$$+ \frac{1}{2}\Big(\frac{1-v^2}{1+v^2}\Big)^2\Big(\vec{\theta}-\frac{\vec{\Delta}}{\omega}\Big)^2\Big] + i\Big(\vec{\theta},\vec{\theta}-\frac{\vec{\Delta}}{\omega}\Big)\Big[\frac{J_0(y_1)J_1(y_2)}{y_2}\Big(\frac{P_2(1-v^2)^2}{2v(1+v^2)x_2^2} + Z\alpha\Big) + \qquad (6)$$

$$+ \frac{J_0(y_2)J_1(y_1)}{y_1}\Big(\frac{P_1(1-v^2)^2}{2v(1+v^2)x_1^2} - Z\alpha\Big)\Big], \qquad y_{1,2} = \theta P_{1,2}/x_{1,2}$$

$$T_2 = \frac{2J_1(y_1)J_1(y_2)}{y_1y_2}\Big\{\frac{\Delta^2}{\omega^2}\frac{P_1P_2}{(x_1x_2)^2} + \frac{2vZ\alpha}{1+v^2}\Big(\frac{P_1}{x_1^2}-\frac{P_2}{x_2^2}\Big)\Big(\frac{\vec{\Delta}}{\omega},\frac{\vec{\Delta}}{\omega}-\vec{\theta}\Big) +$$

$$+ \frac{4v^2(Z\alpha)^2}{(1+v^2)^2}\Big[\frac{2}{\Delta^2}[\vec{\theta}\times\vec{\Delta}]^2 - \Big(\vec{\theta}-\frac{\vec{\Delta}}{\omega}\Big)^2\Big]\Big\}$$

Let us change over to the variables $P_{1,2} \to P_{1,2}(v+\frac{1}{v})/\theta^2$, $x_{1,2} \to x_{1,2}(v+\frac{1}{v})/\theta$ and make the inversion of vector $\vec{\theta}$ $(\theta \to 1/\theta)$. As a result, the Bessel functions become θ - independent and it is easy to take the integral over $d^2\vec{\theta}$. In further calculations of the amplitude M_1 we change over the variables $E = (P_1P_2)^{1/4}$, $t = (P_1/P_2)^{1/2}$, $X = (x_1x_2)^{1/2}/E$, $\tau = \ln(x_1/x_2)$ and then the integral is taken over E and then over X, using the relations (/8/, pp. 732 and 744)

$$\int\limits_0^\infty dx\,x\,e^{icx^2}J_\nu(ax)J_\nu(bx) = \frac{i}{2c}J_\nu\Big(\frac{ab}{2c}\Big)e^{i\pi\nu/2}e^{-i(a^2+b^2)/4c}$$

$$\int\limits_0^\infty dx\,e^{icx}J_0(x) = \frac{i\,\mathrm{sign}(c)}{\sqrt{c^2-1}}, \quad \int\limits_0^\infty dx\,e^{icx}J_1(x) = 1 - \frac{|c|}{\sqrt{c^2-1}}, \quad |c|>1 \qquad (7)$$

and also those obtained by differentiating the expression (7) with respect to the parameter. In calculation of the amplitude M_2 we introduce the variables $E = (p_1 p_2)^{1/2}$, $t = (p_1/p_2)^{1/2}$, $x = (x_1 x_2)^{-1/2}$, $\tau = \ln(x_1/x_2)$ and take the integral over x and then over E , using (7). Finally, one obtains, for the amplitudes

$$M_{1,2} = i\,\frac{32\alpha\omega}{\Delta^2}\int_0^\infty d\tau \int_0^\infty \frac{dv\cdot v}{(1+v^2)^2}\int_0^\infty \frac{dt}{t}\cdot S_{1,2}$$

$$S_1 = \sin^2(Z\alpha\tau)\cdot\frac{B}{\mathcal{D}^3}\left[(1-B)\,\mathrm{ch}\,\tau - d + 2B\left(1-\frac{t^2}{(1+t^2)^2}\right)\left(\frac{3(1-B)d}{\mathcal{D}^2} + 2\,\mathrm{ch}\,\tau\right)\right],$$

$$S_2 = \frac{2(Z\alpha)^2\cos(2Z\alpha\tau)}{d\left(t+\frac{1}{t}\right)^2}\left[e^{-\tau} - \frac{2B}{R} - \frac{gB^2}{R\mathcal{D}(\mathcal{D}+d)}\right] + \frac{3B^2\sin^2(Z\alpha\tau)}{d\,\mathcal{D}^5}(g + d\,\mathrm{ch}\,\tau) +$$

$$+\left[\frac{2\sin^2(Z\alpha\tau)}{d^3} - \frac{Z\alpha\cdot\sin(2Z\alpha\tau)}{d^2}\left(\frac{te^{-\tau}+e^{\tau}/t}{t+1/t}\right)\right]\cdot\left[\frac{1}{R} + \frac{B}{\mathcal{D}}\left(1+\frac{\mathrm{ch}\,\tau}{R}\right) + \frac{gB^2}{\mathcal{D}^3}\right], \qquad (8)$$

$$d = \mathrm{ch}\,\tau + \frac{1}{2}\left(t^2 e^{-\tau} + e^{\tau}/t^2\right), \quad B = \frac{m^2}{\Delta^2}\left(v+\frac{1}{v}\right)^2\left(t + 1/t\right)^2,$$

$$g = (1-B)\,\mathrm{sh}^2\tau - 2d\,\mathrm{ch}\,\tau, \quad \mathcal{D} = \left[d^2 + g(1-B)\right]^{1/2},$$

$$R = \mathcal{D} + d - (1-B)\,\mathrm{ch}\,\tau$$

At $\Delta \gg m$ and $\Delta \ll m$ the asymptotic values of amplitudes derived from expression (8) coincide with the results of Refs. /4-6/. The differential cross section for the unpolarized initial photon is $d\sigma/dx = \frac{m^2}{16\pi\omega^2}(|M_1|^2 + |M_2|^2)$, $x = \Delta^2/m^2$. It is independent on the photon frequency. The contribution of μ - mesons to the amplitudes is obtained from Eq. (8) by means of the substitution $m \to m_\mu$. The quantity $\frac{1}{\sigma_0}\cdot d\sigma/dx$ ($\sigma_0 = (Z\alpha)^4 r_e^2/16\pi$, $r_e = \alpha/m$, $r_e^2/16\pi \simeq 1.58\ \mathrm{mb}$) as a function of the momentum transfer Δ at Z = 1 (curve 1), Z = 47 (curve 2), and Z = 92 (curve 3) is plotted in Fig. 1. The contribution of the μ - mesons is not taken into account in Fig. 1. This contribution becomes noticeable at large Δ so that at

Z = 92 it increases the differential cross section at Δ = 10, 20, 30 MeV by 2, 6.5 and 12.4%, respectively. The dependence of the total cross section σ (the ratio σ/σ_0) on the charge of the Coulomb centre is plotted in Fig. 2. It follows from Figs. 1 and 2 that the Coulomb corrections substantially decreases the cross section. Fig. 3 demonstrates the Δ - dependence of the Stokes parameter ξ_3 , which corresponds to the linear polarization in the scattering plane; $\xi_3 = \frac{2M_1M_2}{|M_1|^2+|M_2|^2}$, $\xi_1 = \xi_2 = 0$. It is seen that the degree of polarization grows with increasing Δ and is Δ - independent in the asymptotics. At a fixed value of Δ , ξ_3 increases with increasing Z .

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Figure captions

Fig. 1. The differential cross section as a function of Δ at $Z = 1$ (curve 1), $Z = 47$ (curve 2), and $Z = 92$ (curve 3).

Fig. 2. The dependence of the total cross section on the charge of the Coulomb centre.

Fig. 3. The Δ - dependence of the degree of linear polarization at $Z = 1$ (curve 1), $Z = 47$ (curve 2), and $Z = 92$ (curve 3).

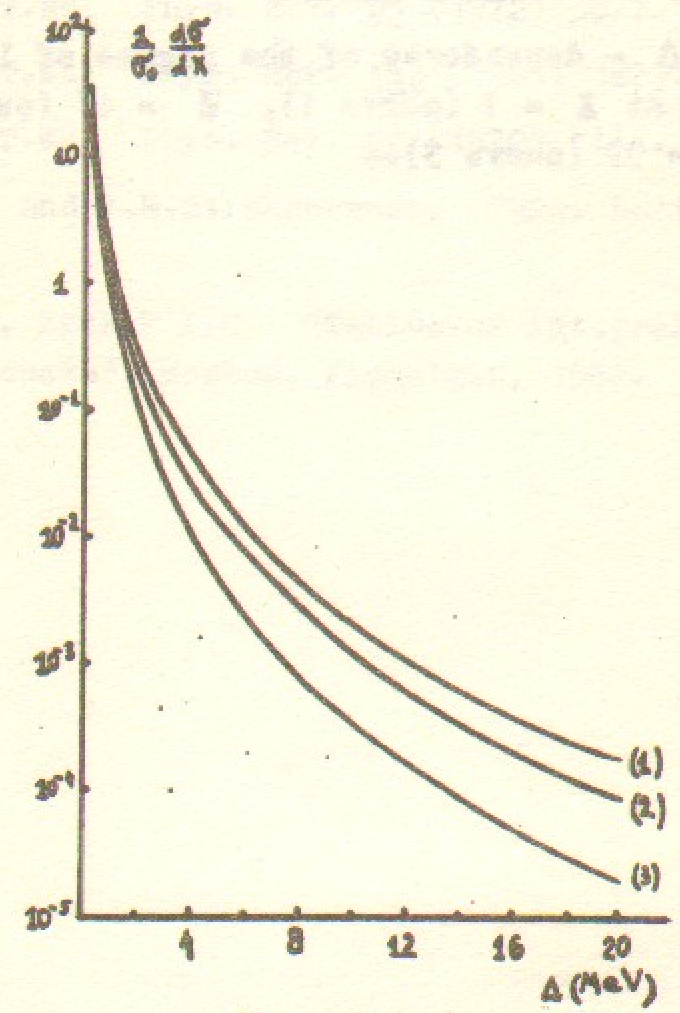
10^{2}
$\frac{1}{\sigma_0}\frac{d\sigma}{d\Delta}$
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
(1)
(2)
(3)
4
8
12
16
20
Δ (MeV)

Fig. 1

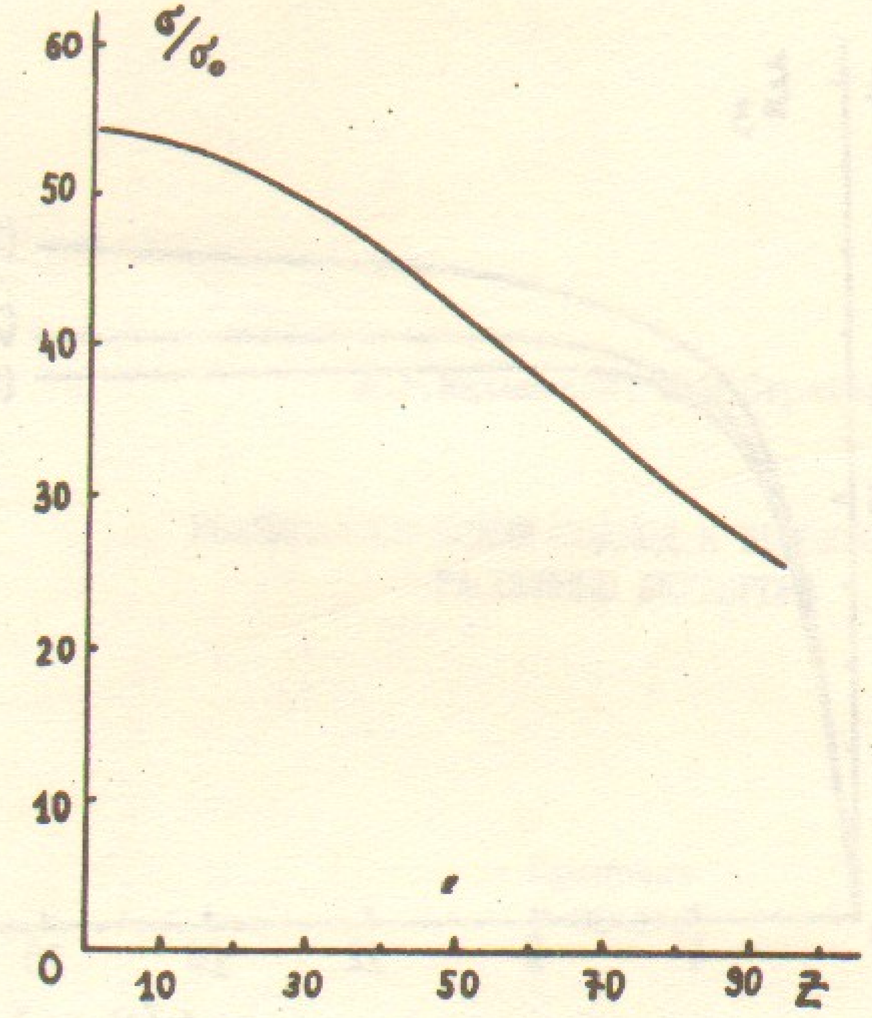
60
σ/σ_0
50
40
30
20
10
0
10
30
50
70
90
Z

Fig. 2

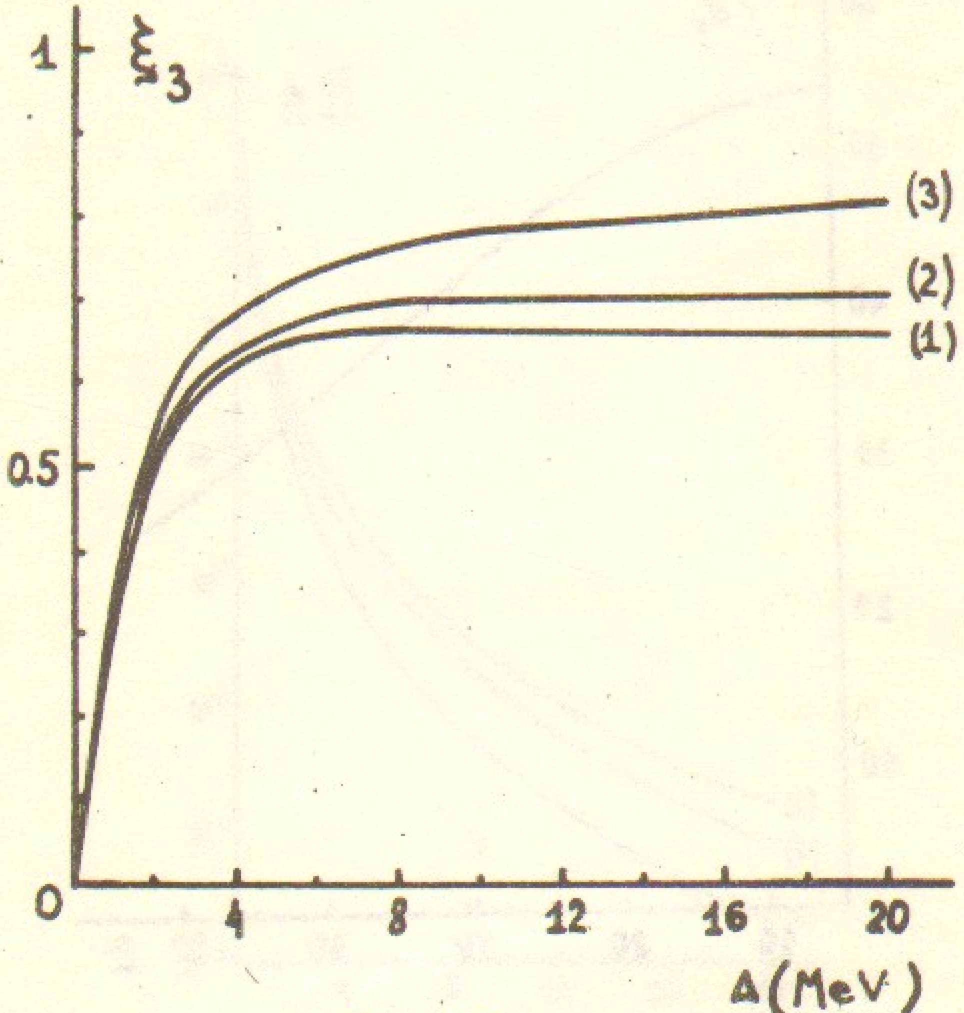

Fig. 3

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КВАЗИКЛАССИЧЕСКИЙ ПОДХОД К ВЫСОКОЭНЕРГЕТИЧЕСКОМУ РАССЕЯНИЮ ДЕЛЬБРЮКА

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