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THE ELECTRON GREEN'S FUNCTION AND THE VACUUM POLARIZATION IN A COULOMB FIELD

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A b s t r a c t

An explicit expression for the renormalized charge density induced in the presence of a Coulomb field is derived both in momentum and coordinate space, by taking advantage of an integral representation for the electron Green's function obtained earlier by the authors.

The accuracy in measurements of energy level differences in muonic atoms (see /1/ and references cited there) makes now the quantum electrodynamical (QED) corrections an observable magnitude. The muon Bohr radius tends to be smaller than*) $\lambda\!\!\!^{-}_c = 1/m$ (m is an electron mass) at large Z (the nuclear charge is $Z|e|$, e is an electron charge, $e = -|e|$) and the effect of vacuum polarization, arising from the coupling of the electron-positron field to the Coulomb field, becomes important. At distances $r \simeq 1/m$, the magnitude of electric field E is $E \simeq 4.4(Z\alpha)10^{13}$ Oe ($\alpha = e^2 = 1/137$ is a constant of a fine structure) and it is the reason why the measurements mentioned above test the QED in the strong field domain. In the theoretical calculation the situation occurs where the fermion propagator is far off the mass shell and cannot be handled in perturbation theory in $Z\alpha$.

The vacuum polarization in a strong Coulomb field has been considered for the first time in /2/, where an expression was obtained for the Laplace transform of r^2 times the vacuum polarization charge density $\rho(r)$. The potential $\varphi_1(r)$, associated with the first (linear in $Z\alpha$) term of expansion of charge density was calculated earlier in /3/. The result of paper /2/ was used in /4/ to obtain the next, proportional to $(Z\alpha)^3$, term of such expansion $\varphi_3(r)$ in coordinate space. The short-distance behaviour of the vacuum polarization potential has been considered in /5/ and /6/ by using formal operator and determinantal techniques. The numerical calculations have been performed of different contributions to the vacuum polarization (see references in /1/). In the present paper an explicit, exact in $Z\alpha$ expression for $\rho(r)$ is obtained directly in coordinate space.

The influence of a Coulomb field on the QED processes is convenient to take into account in the Furry representation. The usual rules of diagram-technique give for $\rho(r)$

$$\tilde{\rho}(r) = -ie \, \underset{x \to x'}{\mathrm{Tr}} [\, G(x,x')\gamma^0 \,] \tag{1}$$

*) The system of units $\hbar = c = 1$ is used.

where γ^0 is the Dirac matrix, tilde denotes unrenormalized quantities. The expression for the electron Green's function in a Coulomb field $G(x,x')$ is of the form

$$G(x,x') = \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t-t')} G(\vec{r},\vec{r}'|\varepsilon) \qquad (2)$$

where, according to the Feinman rules, the contour of integration over ε goes from $-\infty$ to $+\infty$ below the real axis in the left half-plane of variable ε and over it in the right one. We have recently derived /7/ an integral representation for $G(\vec{r},\vec{r}'|\varepsilon)$ which is valid in the whole complex plane ε and does not contain, in contrast to the results of the other papers (see e.g. /8/), the contour integrals. The latter circumstance is convenient in applications.

The known rules (see e.g. /9/) are to be taken into account in the limiting procedure $x \to x'$ in (1). Note that all ambiguities disappear, when renormalization has been performed. Setting $\vec{r} = \vec{r}'$ in formulas (19) and (20) of paper /7/ we have

$$G^{\pm}(\vec{r},\vec{r}|\varepsilon) = \frac{\mp i}{2\pi r^2} \sum_{l=1}^{\infty} l \int_0^{\infty} ds \exp\left\{\pm i\left[2Z\alpha\varepsilon s + 2kr \operatorname{ctg} ks - \pi\nu\right]\right\} T \qquad (3)$$

here

$$T = \frac{x}{2} J'_{2\nu}(x)(\gamma^0\varepsilon + m) \mp iZ\alpha\gamma^0 J_{2\nu}(x)\, k \operatorname{ctg} ks$$

$$\nu = \sqrt{l^2 - (Z\alpha)^2}, \quad x = \frac{2kr}{\sin ks}, \quad k = \sqrt{m^2 - \varepsilon^2} \qquad (4)$$

where $J_{2\nu}(x)$ is the Bessel function. In the case of attractive field; that we consider, G^+ determines the function G in the upper and G^- in the lower half-plane of variable ε. The functions G^+ and G^- coincide, as it should be, on the segment of the real axis $(-m,m)$. It has been shown in /7/ that the derived expression for $G(\vec{r},\vec{r}'|\varepsilon)$ has analytic properties, which follow from the general theory /9/: it has the cuts along the real axis from $-\infty$ to m and from m to ∞ in the complex plane ε, which correspond to the continuous spectrum, and simple poles in the interval $(0,m)$ corresponding to the discrete one. The cited analytic properties and analysis of the expression (3) permit one to deform the contour of integration over

ε in (1) so that it coincides finally with the imaginary axis. After this deformation one can rotate the contour of integration over s in (3) to the imaginary axis: it goes from 0 to $-i\infty$ in the expression G^+ and from 0 to $i\infty$ in G^-. Performing the stated transformations, taking trace and making an obvious replacement of variables, we have

$$\tilde{\rho}(r) = \frac{em^3}{\pi^2 R^2} \sum_{l=1}^{\infty} l \int_0^{\infty} dx \int_0^{\infty} dt\, e^{-y\operatorname{ch} t} f(y,t) \qquad (5)$$

where

$$y = \frac{2bR}{\operatorname{sh} t}, \quad b = \sqrt{x^2+1}, \quad R = mr, \quad \mu = 2Z\alpha\frac{x}{b}$$

$$f(y,t) = y\frac{x}{b}\sin(\mu t)\, I'_{2\nu}(y) - 2Z\alpha \operatorname{cth} t \cos(\mu t)\, I_{2\nu}(y) \qquad (6)$$

here $I_{2\nu}(y)$ is the modified Bessel function of the first kind. Note that expansion of the quantity $\tilde{\rho}(r)$ (5) in $Z\alpha$ contains the odd powers of this parameter only, i.e. the Furry theorem is fulfilled. It is necessary to renormalize the expression for $\tilde{\rho}(r)$ (5). We perform this proceeding from the physically clear requirement: the total induced charge should be equal to zero. It is convenient to perform the renormalization in momentum space. By definition:

$$\tilde{\rho}(\beta) = \int d^3r\, e^{i\vec{q}\vec{r}} \tilde{\rho}(r) = \frac{4\pi}{\beta m^3} \int_0^{\infty} R\,dR\, \tilde{\rho}(r) \sin(\beta R) \qquad (7)$$

where $\beta = \frac{|\vec{q}|}{m}$. Substituting the expression (5) for $\tilde{\rho}(r)$ into eq. (7) and passing to integration over y, instead of R, we get:

$$\tilde{\rho}(\beta) = \frac{4e}{\pi\beta} \sum_{l=1}^{\infty} l \int_0^{\infty} dx \int_0^{\infty} dt \int_0^{\infty} \frac{dy}{y} \sin\left(\frac{\beta y \operatorname{sh} t}{2b}\right) e^{-y\operatorname{ch} t} f(y,t) \qquad (8)$$

The renormalized quantity $\rho(\beta)$ should be equal to zero at the point $\beta = 0$. For this reason we get the renormalized expression for $\rho(\beta)$ if, first, we determine the asymptotic behaviour of the quantity $\tilde{\rho}(\beta)$ at $\beta \to 0$, retaining only the terms, which do not turn into zero at $\beta \to 0$ and, secondly, subtracting these terms from expression (8). The terms of different orders in $Z\alpha$ expansion of $\tilde{\rho}(\beta)$ have different asymptotic behaviour in the limit $\beta \to 0$. So, the linear term $\rho_1(\beta)$, which can be obtained by changing $f(y,t) \to f_1(y,t)$ in the relation (8), where

$$f_1(y,t) = 2Z\alpha\left[yt\left(\frac{x}{\ell}\right)^2 I'_{2\ell}(y) - \coth t\, I_{2\ell}(y)\right] \tag{9}$$

containes the term $C_1/\beta^2 + C_2$ at $\beta \to 0$. The quantity $\widetilde{\rho}_1(\beta)$ corresponds in terms of the theory of perturbation to the diagram of polarization operator of the lowest order in α:

$$\widetilde{\rho}_1(\beta) = \frac{Ze}{|\vec{q}|^2}\,\widetilde{\mathcal{P}}(-|\vec{q}|^2) \tag{10}$$

and subtraction of the term $C_1/\beta^2 + C_2$ coincides with the usual procedure of the renormalization of the polarization operator $\widetilde{\mathcal{P}}$. After subtraction we get

$$\rho_1(\beta) = \frac{e(Z\alpha)}{3\pi}\left[\frac{5}{3} - \frac{4}{\beta^2} + \left(\frac{2}{\beta^2}-1\right)\sqrt{1+\frac{4}{\beta^2}}\,\ln\left(\frac{\sqrt{1+\frac{4}{\beta^2}}+1}{\sqrt{1+\frac{4}{\beta^2}}-1}\right)\right] \tag{11}$$

for the renormalized quantity $\rho_1(\beta)$. The expression (11) coincides, as it should be, (in sense of relation (10), where tildes are to be removed) with the result of direct calculation of polarization operator (see /9/). The potential corresponding to charge density ρ_1 is known as the Uehling potential /3/. It has the form

$$\varphi_1(r) = -\frac{2e}{3\pi r}Z\alpha\left[K_0(2R) - \frac{1}{2}\int_1^\infty \frac{dx}{x^4}\,\frac{x^2+1}{\sqrt{x^2-1}}\,e^{-2Rx}\right] \tag{12}$$

here $K_0(2R)$ is the modified Bessel function of the third kind. We write the Uehling potential $\varphi_1(r)$ in the form (12), which makes obvious its behaviour at $R \to 0$.

The terms of the order $(Z\alpha)^3$ in (8) correspond to the diagrams of the type of light by light scattering. These give the term $C_3 + C_4 \ln\beta$ in the limit $\beta \to 0$ - the fact was noted in /2/. At last, the terms of order $(Z\alpha)^5$ and higher in (8) have a constant limit at $\beta \to 0$. It's easy to calculate this constant by changing in (8) $\sin(\beta y\, \mathrm{sh}\, t/2\ell) \to (\beta y\, \mathrm{sh}\, t/2\ell)$. Performing subtractions, we obtain finally the renormalized expression for $\rho(\beta) = \rho_1(\beta) + \rho_2(\beta)$, where $\rho_1(\beta)$ is

defined in (11) and for $\rho_2(\beta)$ we have:

$$\rho_2(\beta) = \frac{4e}{\pi}\left\{\frac{1}{\beta}\sum_{\ell=1}^{\infty}\ell\int_0^\infty dx\int_0^\infty dt\int_0^\infty\frac{dy}{y}\left[\sin\left(\frac{\beta y\,\mathrm{sh}\,t}{2\ell}\right)e^{-y\,\mathrm{ch}\,t}\Big(f(y,t) - f_1(y,t)\Big) + \sin\left(\frac{\beta y t}{\ell}\right)g(y,t)\right] + \Omega\right\} \tag{13}$$

where

$$g(y,t) = \frac{(Z\alpha)^3}{\sqrt{\pi y}}\exp\left(-yt^2 - \frac{\ell^2}{y}\right)\left[\frac{1}{yt} - 4t\,\frac{x^2}{\ell^2} + \frac{4}{3}yt^3\,\frac{x^4}{\ell^4}\right] \tag{14}$$

$$\Omega = \sum_{\ell=1}^{\infty}\ell\,\mathrm{Im}\left\{\ln\Gamma(\nu - iZ\alpha) + \frac{1}{2}\ln(\nu - iZ\alpha) + iZ\alpha\,\psi(\ell) + \frac{iZ\alpha}{2\ell} - i\,\frac{(Z\alpha)^3}{3\ell^2}\right\}$$

here $\psi(t) = \frac{d}{dt}\ln\Gamma(t)$. We get the renormalized expression for the induced charge density in coordinate space $\rho_2(r)$, performing the inverse Fourier transformation:

$$\rho_2(\vec{r}) = \frac{em^3}{\pi^2R^2}\sum_{\ell=1}^{\infty}\ell\int_0^\infty dx\int_0^\infty dt\left[e^{-y\,\mathrm{ch}\,t}\Big(f(y,t) - f_1(y,t)\Big) + g(y_1,t)\right] + \frac{4e}{\pi}\,\Omega\,\delta(\vec{r}) \tag{15}$$

here $y = 2\ell R/\mathrm{sh}\,t$, $y_1 = \ell R/t$, the other quantities are determined in eqs. (6), (9), (14). The explicit expression for $\rho(r)$ enables one to write down a set of expressions for corresponding potential $\varphi(r)$, taking into account the fact, that the total induced charge is equal to zero. As mentioned above, the behaviour of the potential $\varphi(r)$, at small R, is of importance in some cases. We give a few first terms of expansion of the potential $\varphi(r)$ at $R \equiv mr \ll 1$

$$\varphi(r) = em\left[-\frac{2Z\alpha}{3\pi}\left(\ln\frac{1}{R} - C - \frac{5}{6}\right) + \frac{A}{R} + B + FR + \mathcal{D}R^{2\nu_1} + \dots\right] \tag{16}$$

where $C = 0.577\ldots$ is Euler's constant, $\nu_1 = \sqrt{1-(Z\alpha)^2}$

$$A = \frac{4}{\pi}\sum_{\ell=1}^{\infty}\ell\,\mathrm{Im}\left[\ln\Gamma(\nu - iZ\alpha) + \frac{1}{2}\ln(\nu - iZ\alpha) - (\nu - iZ\alpha)\psi(\nu - iZ\alpha) + \frac{iZ\alpha}{2\ell} - iZ\alpha\,\ell\,\psi'(\ell)\right]$$

$$F = \frac{4}{\pi}Z\alpha\sum_{\ell=1}^{\infty}\frac{\ell}{\nu}\,\mathrm{Re}\left\{\frac{\ell^2\psi'(\nu - iZ\alpha)}{4\nu^2-1} - \frac{\ell^2+\nu}{2\ell^2(2\nu+1)}\right\}$$

$$B = -2Z\alpha\sum_{\ell=1}^{\infty}\frac{\ell}{\nu}\int_0^\infty dt\,J_0(2Z\alpha t)\,e^{-2\nu t}\left(\coth t - \frac{t}{\mathrm{sh}^2 t}\right) \tag{17}$$

$$\mathcal{D} = -\frac{8Z\alpha}{(4\nu_1^2-1)}\,\frac{\Gamma(3/2-\nu_1)}{\nu_1^2\,\Gamma(2\nu_1+2)\,\Gamma(1/2)}\int_0^\infty dt\,t\left(\frac{Z\alpha t}{\mathrm{sh}^2 t}\right)^{\nu_1}J_{-\nu_1}(2Z\alpha t)$$

here J_{ν_1}, $J_{-\nu_1}$ are the Bessel functions, $\psi'(t) = \frac{d\psi}{dt} = \frac{d^2}{dt^2}\ln\Gamma(t)$. The quantity Ae corresponds to an induced point charge ($\delta Q'$ in notation of papers /2/ and /5/) at the origin and contains the terms $(Z\alpha)^3$ and higher. The quantity $\delta Q'$ was at first obtained in paper /2/ in the form, different from (17). It was recalculated in paper /5/, which is entirely devoted to this question, in the form coinciding with our result for Ae in eq. (17). This form agrees, to all orders, of $Z\alpha$ with the result of paper /2/, as it was shown in /5/. The quantity $\delta Q'$ coincides with the limit of $\rho_2(\beta)$ in eq. (13) at $\beta \to \infty$. Within our approach a part of $\delta Q'$ has arisen at renormalization (Ω in eq. (13)) and the region of $x \gg 1$ gives the contribution to $\delta Q'$ in the remaining integral. So, we can neglect unity in expression for $b = \sqrt{x^2+1}$, that corresponds to zero-electron-mass limit, utilized in paper /5/. The paper /6/ is devoted to calculation of the coefficients F and D. The coefficient F in eq. (17) coincides with the corresponding result of /6/. The coefficient D is presented in /6/ in a rather complicated form. The first two terms of expansion in Z of the quantity D in eq. (17) agree with the corresponding terms in /6/. The coefficient D in eq. (17) has a singularity (pole) at $\nu_1 = 1/2$, which cancels in eq. (16) with the same singularity in the coefficient F (at $\nu_1 \to 1/2$, $R^{2\nu_1} \to R$) this question was discussed in details in /6/. The coefficients B, F and D have also singularities $\sim 1/\nu_1$ at $\nu_1 \to 0$ ($Z\alpha \to 1$). These singularities cancel in the expression for $\varphi(r)$ (eq. (16)) in pairs: B and $DR^{2\nu_1}$, FR and a term omitted in eq. (16), which is proportional to $R^{2\nu_1+1}$, that we have established directly. The term $\sim (Z\alpha)^3$ in expansion of quantity B was obtained in paper /4/ and is in agreement with our result.

At large $R \gg 1$ the Uehling potential $\varphi_1(r)$ decreases exponentially. The leading term of the potential $\varphi_2(r)$ (corresponding to $\rho_2(r)$ (15)) is proportional to r^{-5} and only $(Z\alpha)^3$ term survives, that we have checked by direct calculation. This agrees with the result of paper /2/ and may be easily interpreted in terms of effective Lagrangian (see /2/, Appendix III).

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ФУНКЦИЯ ГРИНА ЭЛЕКТРОНА В КУЛОНОВСКОМ ПОЛЕ
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