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THE O(2.1) ALGEBRA AND THE ELECTRON GREEN's FUNCTION IN A COULOMB FIELD

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Abstract

An integral representation for the Green's function of a charged particle moving in a Coulomb field is obtained, with the help of the O(2.1) algebra, in the form convenient for applications.

The influence of a Coulomb field on the QED processes is convenient to take into account in the Furry representation. In this case, it is necessary to have an explicit expression for the Green's function $G(x,x')$ for a charged particle in this field. The function $G(x,x')$ is time-dependent in the combination $t-t'$:

$$G(x,x') = \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon(t-t')} \; G(\vec{r},\vec{r}\,'|\varepsilon) \qquad (1)$$

It follows from the general theory (see, e.g. Ref. /1/) that the function $G(\vec{r},\vec{r}\,'|\varepsilon)$ has, in the complex plane ε , the cuts along the real axis from $-\infty$ to m and from m to ∞ , which correspond to the continuous spectrum, and also has the simple poles in the interval (0, m) for the attraction field and in the interval (-m,0) for the repulsion field*. According to the Feinman rules, the contour of integration over ε in (1) goes from $-\infty$ to ∞ below the real axis in the left half-plane and over it in the right one.

In the paper /2/ the Green's function in a Coulomb field was derived by using the explicit form of expansion $G(\vec{r},\vec{r}\,'|\varepsilon)$ with respect to the eigenfunctions of the corresponding wave equation. In that paper the integral representation for the function $G(\vec{r},\vec{r}\,'|\varepsilon)$ contains the contour integral, that complicates its use in applications.

In the present paper a representation for $G(\vec{r},\vec{r}\,'|\varepsilon)$ is obtained by means of the operator method. This representation is valid in the whole complex plane ε , and does not contain the contour integrals. It is worth noting that the operator method used does not require the knowledge of the solutions to the wave equation.

Let us now consider the function $G(\vec{r},\vec{r}\,'|\varepsilon)$ for real ε in the interval $(-m, m)$. We shall represent it in the form

$$G(\vec{r},\vec{r}\,'|\varepsilon) = \frac{1}{\hat{\mathcal{P}} - m} \, \delta(\vec{r}-\vec{r}\,') \qquad (2)$$

where $\hat{\mathcal{P}} = \gamma^0(\varepsilon + \frac{Z\alpha}{r}) - \vec{\gamma}\vec{p}$, $\vec{p} = -i\vec{\nabla}$; $Z>0$ for the attraction field, $\alpha = 1/137$ is the constant of a fine structure; the standard representation for γ - matrices is used /1/. We shall rewrite

* The system of units $\hbar = c = 1$ is used.

the operator $(\hat{p}-m)^{-1}$ in the form $(\hat{p}+m)[r(\hat{p}^2-m^2)]^{-1}r$; then eq. (2) may be of the following form:

$$G(\vec{r},\vec{r}\,'|\varepsilon)=-i(\hat{p}+m)\int_0^\infty ds\, e^{2iZ\alpha\varepsilon s}\, e^{-i[rP_r^2+r(m^2-\varepsilon^2)+\frac{K}{r}]s}\,\frac{\delta(r-r')}{r'}\,\delta(\vec{n}-\vec{n}\,') \qquad (3)$$

where $P_r=-\frac{i}{r}\frac{\partial}{\partial r}r$, $K=\vec{L}^2-iZ\alpha\gamma^0\vec{\gamma}\vec{n}-(Z\alpha)^2$, $\vec{L}$ is the orbital angular momentum operator; the representation $\delta(\vec{r}-\vec{r}\,')=\frac{1}{rr'}\delta(r-r')\delta(\vec{n}-\vec{n}\,')$, $\vec{n}=\vec{r}/|\vec{r}|$, $\vec{n}\,'=\vec{r}\,'/|\vec{r}\,'|$ is used. It is easy to find the eigenfunctions of operator K and construct from them the corresponding projection operator (see Ref. /3/), which satisfies the relations

$$KP_\lambda(\vec{n},\vec{n}\,')=\lambda(\lambda+1)P_\lambda(\vec{n},\vec{n}\,');\ \sum_\lambda P_\lambda(\vec{n},\vec{n}\,')=\delta(\vec{n}-\vec{n}\,');\ \int d\vec{n}\,'P_\lambda(\vec{n},\vec{n}\,')P_{\lambda'}(\vec{n}\,',\vec{n}\,'')=\delta_{\lambda\lambda'}P_\lambda(\vec{n},\vec{n}\,'') \qquad (4)$$

where $\lambda=\pm\sqrt{(j+1/2)^2-(Z\alpha)^2}$, j is the value of the total angular momentum; summation over λ means that over j and over two signs in the definition of λ . Write down an explicit form of the projection operator:

$$P_\lambda(\vec{n},\vec{n}\,')=\sum_{m,\mu}\begin{pmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}$$

$$P_{11}=P_{22}=\frac{\lambda+\varkappa}{2\lambda}\Omega_{jlm}(\vec{n})\Omega^+_{jlm}(\vec{n}\,') \qquad (5)$$

$$P_{12}=P_{21}=\frac{\mu Z\alpha}{2\lambda}\Omega_{jlm}(\vec{n})\Omega^+_{jl'm}(\vec{n}\,')$$

where $\Omega_{jlm}(\vec{n})$ are the simultaneous eigenfunctions of $\vec{J}^2$, $\vec{L}^2$ and j_z (see Ref. /1/), $l=j+\frac{\mu}{2}$, $l'=j-\frac{\mu}{2}$, $\varkappa=\mu(j+\frac{1}{2})$, $\mu=\pm1$. If one substitutes the expression for $\delta(\vec{n}-\vec{n}\,')$ from eq. (4) into eq. (3), then the problem is reduced to the calculation of the action of the operator $\exp[-2is(T_1+\frac{k^2}{2}T_3)$ on $\delta(r-r')$. Here $k^2=m^2-\varepsilon^2$ and the operators

$$T_1=\frac{1}{2}\left[rP_r^2+\frac{\lambda(\lambda+1)}{r}\right]\ ;\quad T_3=r \qquad (6)$$

are introduced. The commutation relations for them and for the operator $T_2=rP_r$

$$[T_1,T_2]=-iT_1\ ,\ [T_1,T_3]=-iT_2\ ,\ [T_2,T_3]=-iT_3 \qquad (7)$$

are equivalent to that of the O(2.1) algebra generators. It is just the availability of relations (7) that allows one to solve the problem stated in closed form*. Let us first calculate a result of the action of operator $\exp(-iuT_1)$ on the function $f(r)$ which admits the Laplace transformation:

$$f(r)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\sigma\, F(\sigma)\, e^{\sigma r}r^\delta\ ,\quad F(\sigma)=\int_0^\infty dr\, f(r)r^{-\delta}e^{-\sigma r} \qquad (8)$$

δ is found from the condition $T_1r^\delta=0$. Using eq. (6), we choose

$$\delta=\begin{cases}\lambda & \text{at } \lambda>0\\ |\lambda|-1 & \text{at } \lambda<0\end{cases} \qquad (9)$$

According to (8), we have to effect the function r^δ with the operator $\exp(-iuT_1)\exp(\sigma T_3)$; to accomplish this, let us represent this operator as follows:

$$e^{-iuT_1}e^{\sigma T_3}=e^{-iaT_3}e^{-ibT_2}e^{-icT_1} \qquad (10)$$

where a , b and c are the functions of u and σ. Let us differentiate both sides of eq. (10) with respect to u . With the use of relations (7) one can displace all exponents to the right and to cancel them in both sides of the equality. Finally, equalling the coefficients at operators T_1, T_2 and T_3 , we obtain a system of differential equations for a , b and c , the solution to which looks as follows:

$$a=\frac{i\sigma}{1-\frac{i\sigma u}{2}}\ ;\quad b=2\ln\left(1-\frac{i\sigma u}{2}\right);\quad c=\frac{u}{1-\frac{i\sigma u}{2}} \qquad (11)$$

if the obvious boundary conditions are taken into account. Since

$$e^{-ibT_2}\varphi(r)=e^{-b}\varphi(re^{-b}) \qquad (12)$$

where $\varphi(r)$ is an arbitrary function, and using eqs. (10), (11) and (12), one obtains

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* The papers /4/ and /5/ give some examples of applying the O(2.1) algebra in a Coulomb field and show how the use of this algebra makes it possible to find the energy spectrum in a Coulomb field. In the paper /6/ this algebra was used to find the Green's function in operator form.

$$e^{-iuT_1} e^{\sigma T_3} z^{\delta} = \frac{z^{\delta}}{\left(1-\frac{i\sigma u}{2}\right)^{2(\delta+1)}} \exp\left(\frac{\sigma z}{1-\frac{i\sigma u}{2}}\right) \qquad (13)$$

Relation (13) enables one to write the result of the action of the operator $\exp(-iuT_1)$ on $f(z)$, represented in the form (8); After that, the integral over σ is taken, and finally, we have

$$e^{-iuT_1} f(z) = -\frac{2}{u}\int_0^{\infty} dx\, f(x)\sqrt{\frac{x}{z}}\, J_{2\delta+1}\left(\frac{4}{u}\sqrt{xz}\right) e^{\frac{2i}{u}(x+z)} e^{-i\pi\delta} \qquad (14)$$

where $J_{2\delta+1}$ is the Bessel function. Acting in the same way as in derivation of (10) and (11), we write the operator of interest $\exp[-2is(T_1+\frac{k^2}{2}T_3)]$ in the form

$$e^{-2is\left(T_1+\frac{k^2}{2}T_3\right)} = e^{-ia_1 T_3} e^{-ib_1 T_2} e^{-ic_1 T_1} \qquad (15)$$

where $a_1 = k\,tg(ks)$, $b_1 = 2\ln[\cos(ks)]$, $c_1 = \frac{2}{k}tg(ks)$. Using (14) with $f(z)=\delta(z-z')$, $u=c_1$ and (12), we find from eq. (3)

$$G(\vec{z},\vec{z}'|\varepsilon) = i(\hat{P}+m)\sum_{\lambda}\int_0^{\infty}\frac{ds}{\sqrt{zz'}}\frac{k e^{-i\pi\delta}}{\sin ks} J_{2\delta+1}\left(\frac{2k\sqrt{zz'}}{\sin ks}\right) P_{\lambda}(\vec{n},\vec{n}')\, e^{ik(z+z')ctg\,ks+2iz\alpha\varepsilon s} \qquad (16)$$

For the purpose of further transformations of (16) it's convenient to represent $\vec{\gamma}\vec{P}$ as follows:

$$\vec{\gamma}\vec{P} = [P_z - \frac{i}{z}(1+\vec{\Sigma}\vec{L})]\vec{\gamma}\vec{n} \qquad (17)$$

where $\vec{\Sigma} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$, $\vec{\sigma}$ are Pauli matrices. Using the recurrence relations for the Bessel functions and the summation formula

$$\sum_m \Omega_{j,j\pm 1/2,m}(\vec{n})\Omega^{+}_{j,j\pm 1/2,m}(\vec{n}') = \frac{1}{4\pi}[j+1/2 \mp \vec{\sigma}\vec{L}] P_{j\pm 1/2}(\vec{n}\vec{n}') \qquad (18)$$

where $P_n(x)$ are the Legendre polynomials, we obtain for $G(\vec{z},\vec{z}'|\varepsilon)$

$$G(\vec{z},\vec{z}'|\varepsilon) = -\frac{i}{4\pi zz'}\sum_{\ell=1}^{\infty}\int_0^{\infty} ds\, \exp\{i[2z\alpha\varepsilon s + k(z+z')ctg\,ks - \pi\nu]\}\, T(s) \qquad (19)$$

where

$$T(s) = [1+\vec{n}\vec{n}'+i\vec{\Sigma}[\vec{n}\times\vec{n}']][\frac{y}{2}J'_{2\nu}(y)(\gamma^0\varepsilon+m) - iz\alpha J_{2\nu}(y)\gamma^0 k\, ctg\,ks]B +$$

$$+[1-\vec{n}\vec{n}'-i\vec{\Sigma}[\vec{n}\times\vec{n}']](\gamma^0\varepsilon+m)\ell J_{2\nu}(y)A + imz\alpha\gamma^0(\vec{\gamma},\vec{n}+\vec{n}')J_{2\nu}(y)B +$$

$$+\left[\frac{ik^2(z-z')}{2\sin^2(ks)}(\vec{\gamma},\vec{n}+\vec{n}')B - \ell k\, ctg(ks)(\vec{\gamma},\vec{n}-\vec{n}')A\right] J_{2\nu}(y) \; ; \qquad (20)$$

$$\nu = \sqrt{\ell^2-(z\alpha)^2}\;;\; A = \frac{d}{dx}[P_\ell(x)+P_{\ell-1}(x)]\;;\; B = \frac{d}{dx}[P_\ell(x)-P_{\ell-1}(x)]$$

$$x = \vec{n}\vec{n}'\;;\quad y = \frac{2k\sqrt{zz'}}{\sin(ks)}\;,\quad k = \sqrt{m^2-\varepsilon^2}$$

Recall that so far we have regarded ε as a real quantity in the interval $(-m, m)$. The analysis shows that formula (19) gives, for $z>0$, a direct analytic continuation to the upper (lower for $z<0$) half-plane of the variable ε. Denote this function by $G^{(+)}(\vec{z},\vec{z}'|\varepsilon)$ and the analytic continuation of $G(\vec{z},\vec{z}'|\varepsilon)$ to the lower (upper for $z<0$) half-plane ε by $G^{(-)}(\vec{z},\vec{z}'|\varepsilon)$. For the latter we have

$$G^{(-)}(\vec{z},\vec{z}'|\varepsilon) = \frac{i}{4\pi zz'}\sum_{\ell=1}^{\infty}\int_0^{\infty} ds\, \exp\{-i[2z\alpha\varepsilon s + k(z+z')ctg\,ks - \pi\nu]\}\, \tilde{T}(s) \qquad (21)$$

where $\tilde{T}(s)$ is obtained from $T(s)$, by alternating the sign of $ctg(ks)$. This corresponds to the integration over s in formula (3) from 0 to $-\infty$. The functions $G^{(+)}(\vec{z},\vec{z}'|\varepsilon)$ (19) and $G^{(-)}(\vec{z},\vec{z}'|\varepsilon)$ (21) coincide, as it should be, on the segment of the real axis $(-m, m)$. In this range of values of ε the quantity k is real. In integrating over s, the Bessel function passes from sheet to sheet at points $s=\pi n/k$ at integer $n = 1,2\ldots$, acquiring the phase $e^{-2i\pi\nu}$. We decompose the integral over s into the sum of integrals over segments of length π/k. Using the phase relations for the Bessel function and the periodicity of $ctg(ks)$, we obtain a representation for $G(\vec{z},\vec{z}'|\varepsilon)$ in the domain $-m<\varepsilon<m$:

$$G(\vec{z},\vec{z}'|\varepsilon) = \frac{1}{8\pi zz'k}\cdot\frac{1}{\sin[\pi(\frac{z\alpha\varepsilon}{k}-\nu)]}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\tau\, \exp\{i[k(z+z')tg\,\tau - \frac{2z\alpha\varepsilon}{k}\tau]\}\, T\left(\frac{\pi-2\tau}{2k}\right) \qquad (22)$$

The denominator in (22) tends to ∞ at points $\frac{Z\alpha\varepsilon}{k} - \nu = n$ for any integer n. However, the integral over τ also vanishes for negative n at these points so that the expression (22) has the poles only for $n = 0,1,2, \ldots$. Since ν is positive, this condition may be satisfied only at $Z\varepsilon > 0$. Hence, the simple poles corresponding to the discrete spectrum, are at the points

$$\varepsilon = \frac{mZ}{|Z|}\left[1 + \left(\frac{Z\alpha}{n+\nu}\right)^2\right]^{-1/2}$$

On the segments of the real axis $(-\infty, m)$ and (m, ∞) the functions $G^{(+)}$ and $G^{(-)}$ are different, being the values of the same function at different sides of the cut.

Quite similar are the calculations for the case of a particle with spin 0, the wave function of which satisfies the Klein-Gordon equation. We shall give the final result:

$$G_0^{(\pm)}(\vec{r},\vec{r}\,'|\varepsilon) = -\frac{1}{4\pi\sqrt{rr'}}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\vec{n}\vec{n}\,')\int_0^\infty \frac{k\,ds}{\sin(ks)}\exp\left\{\pm i\left[2Z\alpha\varepsilon s + k(r+r')\operatorname{ctg} ks - \pi\mu\right]\right\} J_{2\mu}(y) \tag{23}$$

where $\mu = \sqrt{(\ell+1/2)^2 - (Z\alpha)^2}$, the remaining notation and the sign $\pm$ have the same sense as in the preceding part of the paper. The analytic properties of the function G_0 are analogous to those of the function G. The nonrelativistic limit is derived from eq. (23) by substitutions: $\mu \to \ell + 1/2$, $k \to \sqrt{-2mE}$ where E is the nonrelativistic energy ($E = \varepsilon - m$), $\varepsilon \to m$. In addition, according to the normalization of the nonrelativistic Green's function, eq. (23) should be multiplied by $2m$. As soon as the substitutions are made, it is possible to carry out the summation over ℓ by using the formula /7/

$$\sum_{\ell=0}^{\infty}(-1)^\ell(2\ell+1)P_\ell(\cos\varphi)J_{2\ell+1}(x) = \frac{x}{2}J_0\left(x\cos\frac{\varphi}{2}\right) \tag{24}$$

As a result, the expression for the nonrelativistic Green's function takes the form

$$G_{nr}^{(\pm)}(\vec{r},\vec{r}\,'|E) = \pm\frac{im}{2\pi}\int_0^\infty \frac{ds\,\varkappa^2}{\sin^2(\varkappa s)}\exp\left\{\pm i\left[2Z\alpha E s + \varkappa(r+r')\operatorname{ctg}\varkappa s\right]\right\} J_0\left(\frac{\varkappa\sqrt{2rr'(1+\vec{n}\vec{n}\,')}}{\sin\varkappa s}\right) \tag{25}$$

here $\varkappa = \sqrt{-2mE}$. The function G_{nr} has the cut along the positive part of the real axis of the complex plane E. For $Z > 0$ (attraction field) it has simple poles at points $E = -\frac{m(Z\alpha)^2}{2n^2}$. At $Z < 0$ there are no poles. To verify this, one can rotate the contour of integration along the imaginary axis. It follows from formula (1) and the properties of the function (25) that $G_{nr}(x,x')$ is different from 0 only at $t > t'$.

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АЛГЕБРА О(2,I) И ФУНКЦИЯ ГРИНА ЭЛЕКТРОНА В КУЛОНОВСКОМ ПОЛЕ

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