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ON THE GENERAL STRUCTURE OF INTEGRABLE EQUATIONS AND THE REDUCTIONS

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1. Introduction

The inverse spectral transform (IST) method, which has been intensively developed in recent years, allows a study of a large number of various partial differential equations (see e.g. /1,2/). One of the basic problems of the IST method is to describe a class of differential equations to which this method is applicable. Another important problem is the reduction problem. A simple and convenient description of the equations integrable by the second-order linear spectral problem was presented in /3/. The method suggested in /3/ (the AKNS method) has been generalized to the matrix spectral problem of arbitrary order /4-9/ and to a number of other spectral problems /10-13/. The reduction problem has been comprehensively studied in recent time /14-17/.

In the present paper we are going to consider both these problems (the problem of description of integrable equations and the reduction problem) for an arbitrary-order linear matrix bundle:

$$\frac{1}{i}\frac{\partial \Psi}{\partial x} = \left(\lambda A + P(x,t)\right)\Psi \qquad (1.1)$$

where A is an arbitrary regular matrix (i.e. all its eigenvalues are different), $P(x,t)$ is the N x N matrix, such that $\lim_{|x|\to\infty} P(x,t) = P_\infty$, where P_∞ is an arbitrary constant matrix.

We shall construct an infinite Abelian group of transformations $P \to P'$ connected with the bundle (1.1) and show that the general form of equations integrable with the help of (1.1) is as follows:

$$\frac{\partial P(x,t)}{\partial t} + i\sum_{n=1}^{N-1}\sum_{m=0}^{n}\Omega_n\left(L_A^+,t\right)\left(L_A^+\right)^m\left[\left(T_{nm}\right)_{0(A)}, P\right] = 0 \qquad (1.2)$$

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* It is noteworth that in Refs. /3-13/ it is assumed that $P_\infty = 0$. Generalization to the case $P_\infty \neq 0$ is a non-trivial one.

where $\Omega_n(\lambda, t)$ are arbitrary functions which are meromorphic with respect to λ ; $T_{nm} = \frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}(\lambda A + P_\infty)^n\big|_{\lambda=0}$; $(T_{nm})_{0(A)}$ stands for the projection of the matrix T_{nm} onto the zero component of the Fitting decomposition (see below); $L_A^+\varphi = -L^+\varphi_A$ where $[A, \varphi_A] \stackrel{def}{=} \varphi$ and the operator L^+ is

$$L^+ = i\frac{\partial}{\partial x} + [P(x), \cdot\,]_{F(A)} + i\left[P(x), \int_{-\infty}^{x} dy\, [P(y), \cdot\,]_{0(A)}\right].$$

We shall show that all equations of the form (1.2) are Hamiltonian ones with respect to the infinite family of Hamiltonian structures. The explicit form of Poisson brackets, symplectic forms and Hamiltonians is given. Equations (1.2) are Lagrangian ones as well. We are going to examine the transformation properties of equations (1.2), in particular, to describe their groups of symmetry.

The reduction problem will be analysed. The specific feature of this problem within the frame of the AKNS method lies in a possibility of enumerating the functions $\Omega_n(\lambda, t)$ for which equations (1.2) permits a definite reduction. As the examples, we shall consider the Z_N and $\tilde{Z}_N$ reductions, the $P^T = -P$ reduction and the others. The general form of equations for these reductions is found and their Hamiltonian character is proved. It is shown, in particular, that under $\tilde{Z}_N$ reduction the problem (1.1) is equivalent to the Gelfand-Dikij spectral problem: $\sum_{k=1}^{N} V_k(x)\left(-i\frac{\partial}{\partial x}\right)^k \chi = \lambda^N \chi$.

The paper is organized as follows. The group of transformations connected with the bundle (1.1) is constructed in section 2. The general form of integrable equations is found in section 3. The Hamiltonian and Lagrangian structures of integrable equations are considered in section 4. The classic r - matrix is calculated in this section as well. The transformation properties of integrable equations are briefly discussed in the next, fifth section. Section 6 is devoted to the reduction problem within the framework of the AKNS method. The reductions Z_N and $\tilde{Z}_N$ and the other linear reductions are considered in the seventh, eighth and ninth sections.

II. Construction of the transformation group connected with the bundle (1.1)

In an analysis of the transformations connected with the bundle (1.1) the so-called Fitting decomposition of the matrix algebra $gl(N,C)$ with respect to A plays a significant role. It is the decomposition into the direct sum: $gl(N,C) = g_{0(A)} \oplus g_{F(A)}$ where $g_{0(A)}$ is the subalgebra of the matrices commuting with A $(g_{0(A)} = \{g \in gl(N,C), [g, A] = 0\})$ and $g_{F(A)}$ is a direct sum of non-zero root subspaces (see, e.g. /18,19/). Recall some properties of the Fitting decomposition /18,19/: the subspace $g_{0(A)}$ is the Lie subalgebra ($[g_{0(A)}, g_{0(A)}] \subset g_{0(A)}$), the commutator of any element from $g_{0(A)}$ with any element from $g_{F(A)}$ belongs to $g_{F(A)}$: $[g_{0(A)}, g_{F(A)}] \subset g_{F(A)}$. If A is a regular element of $gl(N,C)$ (i.e. all the eigenvalues of the matrix A are different), the subalgebra $g_{0(A)}$ is Abelian, is of dimension N and is the Cartan subalgebra. For an arbitrary matrix φ of order N we have unique decomposition $B = B_{0(A)} + B_{F(A)}$, where $B_{0(A)}$ is the projection of B onto $g_{0(A)}$. and $B_{F(A)}$ is the projection of onto $g_{F(A)}$. These properties of the Fitting decomposition are widely used in all further constructions.

For the potential matrix we have the following decomposition: $P(x,t) = P_{0(A)}(x,t) + P_{F(A)}(x,t)$. A role of the variables $P_{0(A)}(x,t)$ and $P_{F(A)}(x,t)$ in the dynamical systems connected with (1.1) is very different. As is known (see, for example, /20/), the linear spectral problems possess a gauge freedom, that allows various gauge conditions to be imposed on $P(x,t)$. In our case, the problem (1.1) is invariant under the transformations $\Psi(x,t,\lambda) \to \Psi'(x,t,\lambda) = G(x,t)\Psi(x,t,\lambda)$, $P(x,t) \to$

$$P'(x,t) = G(x,t)P(x,t)G^{-1}(x,t) - i\frac{\partial G(x,t)}{\partial x}G^{-1}(x,t)$$

where $G(x,t) = G_{0(A)}(x,t)$. In terms of $P_{0(A)}$ and $P_{F(A)}$, respectively,

$$P_{0(A)}(x,t) \to P'_{0(A)}(x,t) = G(x,t)P_{0(A)}(x,t)G^{-1}(x,t) - i\frac{\partial G(x,t)}{\partial x}G^{-1}(x,t)$$

$$P_{F(A)}(x,t)\to P'_{F(A)}(x,t)=G(x,t)P_{F(A)}(x,t)G^{-1}(x,t).$$

As is easy to see, an appropriate choice of $G(x,t)$ always allows to have $P_{O(A)}=0$. Hence $P_{O(A)}$ is of purely gauge nature, i.e. there is such a matrix $\tilde{G}$ that $P_{O(A)} = i\tilde{G}^{-1}\frac{\partial \tilde{G}}{\partial x}$. Dynamical variables are the variables $P_{F(A)}(x,t)$ only. In view of this, it is natural to consider the gauge $P_{O(A)}=0$. Its meaning is to exclude purely gauge degrees of freedom from $P(x,t)$. This gauge is all the more natural since only $P_{F(A)}$ is reconstructed from the equations of the IST method (for the diagonal matrix A see, e.g. /2/).

Now, we proceed to reduce the transformations in which we are interested. Let us first give some information concerning the direct scattering problem for the bundle (1.1).

We shall assume that the matrix $\tilde{A}=\lambda A+P_\infty$ is regular, i.e. all the eigenvalues μ_i $(i=1,\dots,N)$ of the matrix $\tilde{A}$ are different. Let us introduce, in a standard manner, the fundamental matrix solutions F^+ and F^-, which are given by their asymptotics: $F^+\xrightarrow[x\to+\infty]{}E$, $F^-\xrightarrow[x\to-\infty]{}E$ where E is a fundamental matrix-solution of the system $\frac{\partial E}{\partial x}=i\tilde{A}E$. As is well known (see, e.g. /21/), this system has the infinite set of fundamental solutions. The latter are distinguished by the right multiplication on an arbitrary constant non-degenerated matrix. We shall consider the solution of this system, i.e. the asymptotic of the problem (1.1), of the following form:

$$E(x,\lambda)=\mathcal{D}(\lambda)e^{i\bar{A}x} \qquad (2.1)$$

where $\bar{A}_{ik}=\mu_i\delta_{ik}$ and $\mathcal{D}$ is the matrix diagonalising the matrix $\tilde{A}$ $(\mathcal{D}^{-1}\tilde{A}\mathcal{D}=\bar{A})^*$.

The scattering matrix is also determined in a standard way: $F^+(x,t,\lambda)=F^-(x,t,\lambda)S(\lambda,t)$. The scattering data - the main spectral characteristic of the problem (1.1) - is constructed

* Another asymptotic of the problem (1.1), $E_d=\exp i\tilde{A}x$ is connected with the asymptotic (2.1) as follows: $E_d=E\mathcal{D}^{-1}$.

from the elements of the matrix S. In the compact form the scattering data can be represented as the matrices:

$$R\stackrel{def}{=}S_{F(\tilde{A})}\left(S_{O(\tilde{A})}\right)^{-1} \text{ or } \tilde{R}\stackrel{def}{=}\left(S_{O(\tilde{A})}\right)^{-1}S_{F(\tilde{A})}.$$

Note that the matrices of scattering data have the same number of independent elements as $P(x,t)$ (in the gauge $P_{O(A)}=0$). Transition from one choice of the asymptotics of the problem (1.1) to the other ($E_1\to E_2=E_1K$) leads only to a trivial redefinition of the scattering matrix ($S_1\to S_2=K^{-1}S_1K$).

Let us take now two arbitrary potentials $P(x,t)$ and $P'(x,t)$ (we assume that $\lim_{|x|\to\infty}P(x,t)=\lim_{|x|\to\infty}P'(x,t)=P_\infty$) and two corresponding solutions $\Psi(x,t,\lambda)$ and $\Psi'(x,t,\lambda)$ of the problem (1.1). It is easy to justify that the following relation holds:

$$\Psi'-\Psi=-i\Psi\int_x^\infty dy\,\Psi^{-1}(P'-P)\Psi'. \qquad (2.2)$$

Putting $\Psi=F^+$ in (2.2) and going to the limit $x\to-\infty$, we obtain:

$$S'-S=-iS\int_{-\infty}^{+\infty}dx\,(F^+)^{-1}(P'-P)(F^+)'. \qquad (2.3)$$

Formula (2.3), which relates a change of the potential $P(x,t)$ to a change of the scattering matrix $S(\lambda,t)$, plays a fundamental role in our further constructions.

The mapping $P(x,t)\to S(\lambda,t)$, which is given by the bundle (1.1), determines a correspondence between the transformations $P\to P'$ on the set of potentials $\{P(x,t),\ \lim_{|x|\to\infty}P(x,t)=P_\infty\}$ and the transformations $S\to S'$ on the set of matrices $S(\lambda,t)$. This follows from an obvious diagram

$$\begin{tikzcd} P \arrow[r, "(1.1)"] \arrow[d, "T"'] & S \arrow[d, "T"] \\ P' \arrow[r, "(1.1)"] & S' \end{tikzcd}.$$

Let us consider now the transformations T such that

$$\mathcal{S}(\lambda,t) \xrightarrow{T} \mathcal{S}'(\lambda,t) = B^{-1}(\lambda,t)\,\mathcal{S}(\lambda,t)\,C(\lambda,t) \tag{2.4}$$

where $B(\lambda,t)$ and $C(\lambda,t)$ are arbitrary diagonal matrices, i.e. arbitrary matrices satisfying the conditions $B = B_{O(\bar{A})}$ and $C = C_{O(\bar{A})}$ (recall that $\bar{A}$ is a regular diagonal matrix)*. Under transformations (2.4), for the matrices of scattering data, we have

$$\begin{aligned} R(\lambda,t) &\longrightarrow R'(\lambda,t) = B^{-1}(\lambda,t)R(\lambda,t)B(\lambda,t) \\ \tilde{R}(\lambda,t) &\longrightarrow \tilde{R}'(\lambda,t) = C^{-1}(\lambda,t)\tilde{R}(\lambda,t)\dot{C}(\lambda,t). \end{aligned} \tag{2.5}$$

Thus, we confine ourselves to the transformations $P \to P'$, for which the scattering matrix $\mathcal{S}$ and the matrices of scattering data are transformed in a simple linear way (see (2.4) and (2.5)). Transformations laws (2.4) and (2.5) are a generalization of the basic idea of the IST method. Its essence is to map the nonlinear evolution law of the potential, according to the nonlinear differential equation, into the linear (and readily integrable) evolution law of scattering data (see, e.g. /1,2/).

The main achievement of the generalized AKNS-method consists in constructing, in an explicit form, the transformations of the potential $P \to P'$, which correspond to the transformations $\mathcal{S} \to \mathcal{S}'$ of the form (2.4).

Let us rewrite the relation (2.4) in the form $\mathcal{S}' - \mathcal{S} = (1-B)\mathcal{S}' - \mathcal{S}(1-C)$. From comparison of it with (2.3) we find

$$\left(\mathcal{S}^{-1}(1-B)\mathcal{S}'\right)_{F(\bar{A})} = -i\int_{-\infty}^{+\infty} dx \left((F^+)^{-1}(P'-P)(F^+)'\right)_{F(\bar{A})} \tag{2.6}$$

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* If we choose $E_d = \exp i\bar{A}x$, as the asymptotic of the problem (1.1), then the matrices B and C in the transformations law (2.4) should satisfy the conditions $B = B_{O(\tilde{A})}$ and $C = C_{O(\tilde{A})}$.

where $\varphi_{F(\bar{A})} = \varphi - \varphi_{O(\bar{A})}$, $\left(\varphi_{O(\bar{A})}\right)_{ik} = \varphi_{ii}\delta_{ik}$. $(i,k = 1,\dots,N)$

Note that the following equality

$$\begin{aligned} \left(\mathcal{S}^{-1}(1-B)\mathcal{S}'\right)_{F(\bar{A})} &= -\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}\left((F^+)^{-1}(1-\tilde{B})(F^+)'\right)_{F(\bar{A})} = \\ &= i\int_{-\infty}^{+\infty} dx \left\{(F^+)^{-1}\left((P-P_\infty)(1-\tilde{B}) - (1-\tilde{B})(P'-P_\infty)\right)(F^+)'\right\}_{F(\bar{A})} \end{aligned} \tag{2.7}$$

holds, where $\tilde{B} = \mathcal{D}B\mathcal{D}^{-1}$. Equalizing the right-hand sides of the equalities (2.6) and (2.7), we obtain

$$\int_{-\infty}^{+\infty} dx \left\{(F^+)^{-1}\left(\tilde{B}(P'-P_\infty) - (P-P_\infty)\tilde{B}\right)(F^+)'\right\}_{F(\bar{A})} = 0. \tag{2.8}$$

Writting (2.8) by components and introducing the notation

$\left(\overset{++}{\tilde{\varphi}}{}^{(in)}\right)_{kl} = (F^+)'_{kn}(F^+)^{-1}_{il}$ we have

$$\int_{-\infty}^{+\infty} dx\, tr \left\{\left(\tilde{B}(\lambda,t)(P'(x,t)-P_\infty) - (P(x,t)-P_\infty)\tilde{B}(\lambda,t)\right)\overset{++}{\tilde{\varphi}}{}^{(in)}(x,t,\lambda)\right\} = 0 \quad (i \neq n) \tag{2.9}$$

where tr stands for the matrix trace.

Let us represent now the matrix $B(\lambda,t)$ in the form*

$$B(\lambda,t) = \sum_{n=0}^{N-1} B_n(\lambda,t)\,\bar{A}^n \tag{2.10}$$

where $B_n(\lambda,t)$ $(n = 0,\dots,N-1)$ are the numerical functions, and $\bar{A}^0 \overset{def}{=}$ the identical N x N matrix. Since all the elements of the diagonal matrix $\bar{A}$ are different, any diagonal matrix may be represented in the form (2.10) (see, e.g. /22/, chapter VIII).

Correspondingly, for $\tilde{B}$,

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* Note that in Refs. /7-9/ we have used another representation for arbitrary diagonal matrices.

$$\tilde{B}(\lambda,t)=\sum_{n=0}^{N-1} B_n(\lambda,t)\,\tilde{A}^{n}. \tag{2.11}$$

Using (2.11) and taking into account the equalities

$$\mathrm{tr}(\chi\varphi)=\mathrm{tr}(\chi_{O(A)}\varphi_{O(A)})+\mathrm{tr}(\chi_{F(A)}\varphi_{F(A)})$$

and $$\varphi^{\Delta}_{O(A)}(x)=-i\int_x^{\infty}dy\,\bigl(P'(y)\varphi^{\Delta}_{F(A)}(y)-\varphi^{\Delta}_{F(A)}(y)P(y)\bigr)_{O(A)}$$

(see appendix), one can rewrite the equality (2.9) as follows:

$$\begin{aligned}\int_{-\infty}^{+\infty}dx\,\mathrm{tr}\Bigl\{-i\sum_{n=0}^{N-1}\sum_{m=0}^{n}&\bigl(T_{nm}(P'(x)-P_\infty)-(P(x)-P_\infty)T_{nm}\bigr)_{O(A)}\cdot\\ \cdot\lambda^m B_n(\lambda,t)\int_x^{\infty}dy&\bigl(P'(y)\varphi^{\Delta}_{F(A)}-\varphi^{\Delta}_{F(A)}(y)P(y)\bigr)_{O(A)}+\\ +\sum_{n=0}^{N-1}\sum_{m=0}^{n}&\bigl(T_{nm}(P'(x)-P_\infty)-(P(x)-P_\infty)T_{nm}\bigr)_{F(A)}\lambda^m B_n(\lambda,t)\varphi^{\Delta}_{F(A)}(x)\Bigr\}=0\end{aligned}\tag{2.12}$$

where $$T_{nm}\overset{def}{=}\frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}(\lambda A+P_\infty)^n\Big|_{\lambda=0}.$$

Formula (2.12) contains the product $\lambda\varphi^{\Delta}_{F(A)}(\lambda)$ given in a local way at each point λ of the bundle (1.1). The bundle (1.1) allows to transform this λ - local product into the global one. Indeed, as is shown in Appendix, the relation

$$\Lambda_A\varphi^{\Delta}_{F(A)}(\lambda)=\lambda\varphi^{\Delta}_{F(A)}(\lambda) \tag{2.13}$$

holds where

$$\begin{aligned}\Lambda\varphi=-i\frac{\partial\varphi}{\partial x}&-\bigl(P'(x)\varphi(x)-\varphi(x)P(x)\bigr)_{F(A)}+\\ +iP'(x)\int_x^{\infty}(P'\varphi-\varphi P)_{O(A)}&-i\int_x^{\infty}(P'\varphi-\varphi P)_{O(A)}\cdot P(x)\end{aligned}\tag{2.14}$$

and $\int_x^{\infty} f\overset{def}{=}\int_x^{\infty}dy\,f(y)$, $\Lambda_A\varphi\overset{def}{=}(\Lambda\varphi)_A$.

Hence, for any entire function $B_n(\lambda,t)$,

$$B_n(\lambda,t)\varphi^{\Delta}_{F(A)}(\lambda)=B_n(\Lambda_A,t)\varphi^{\Delta}_{F(A)}(\lambda). \tag{2.15}$$

By virtue of (2.15), the relation (2.12) is equivalent to the following one:

$$\begin{aligned}\int_{-\infty}^{+\infty}dx\,\mathrm{tr}\Bigl\{-i\sum_{n=0}^{N-1}\sum_{m=0}^{n}&\bigl(T_{nm}(P'(x)-P_\infty)-(P(x)-P_\infty)T_{nm}\bigr)_{O(A)}\cdot\\ \cdot\int_x^{\infty}dy\,\bigl(P'(y)(\Lambda_A)^m B_n(\Lambda_A,t)\varphi^{\Delta}_{F(A)}&-(\Lambda_A)^m B_n(\Lambda_A,t)\varphi^{\Delta}_{F(A)}\cdot P(y)\bigr)_{O(A)}+\\ +\sum_{n=0}^{N-1}\sum_{m=0}^{n}\bigl(T_{nm}(P'(x)-P_\infty)&-(P(x)-P_\infty)T_{nm}\bigr)_{F(A)}(\Lambda_A)^m B_n(\Lambda_A,t)\varphi^{\Delta}_{F(A)}\Bigr\}=0\end{aligned}\tag{2.16}$$

Finally, equality (2.16) may be written out as follows:

$$\begin{aligned}\int_{-\infty}^{+\infty}dx\,\mathrm{tr}\Bigl\{\varphi^{\Delta}_{F(A)}(x)\cdot\Bigl\{-i\sum_{n=0}^{N-1}\sum_{m=0}^{n}&B_n(\Lambda^+_A,t)(\Lambda^+_A)^m\bigl\{\mathcal{Y}\bigl(T_{nm}(P'-P_\infty)-\\ -(P-P_\infty)T_{nm}\bigr)_{O(A)}P'(x)-P(x)\,\mathcal{Y}&\bigl(T_{nm}(P'-P_\infty)-(P-P_\infty)T_{nm}\bigr)_{O(A)}\bigr\}+\\ +\sum_{n=0}^{N-1}\sum_{m=0}^{n}B_n(\Lambda^+_A,t)(\Lambda^+_A)^m&\bigl(T_{nm}(P'-P_\infty)-(P-P_\infty)T_{nm}\bigr)_{F(A)}\Bigr\}_{F(A)}\Bigr\}=0\end{aligned}\tag{2.17}$$

where $\Lambda^+_A\varphi\overset{def}{=}(\Lambda_A)^+\varphi=-\Lambda^+\varphi_A$ and the operator Λ^+ is an operator adjoint to the operator Λ with respect to the bilinear form $\langle\psi,\varphi\rangle=\int_{-\infty}^{+\infty}dx\,\mathrm{tr}(\psi_{F(A)}(x)\varphi_{F(A)}(x))$:

$$\begin{aligned}\Lambda^+\varphi=i\frac{\partial\varphi}{\partial x}&-\bigl(\varphi(x)P'(x)-P(x)\varphi(x)\bigr)_{F(A)}+\\ &+i\,\mathcal{Y}(\varphi P'-P\varphi)_{O(A)}\cdot P'(x)-iP(x)\,\mathcal{Y}(\varphi P'-P\varphi)_{O(A)}\end{aligned}\tag{2.18}$$

where $(\mathcal{Y}f)(x)\overset{def}{=}\int_{-\infty}^{x}dy\,f(y)$.

Formula (2.17) is a relation between P, P' and $F^+, F^{+\prime}$ under transformations of the form (2.4). Equality (2.17) is satisfied, if

$$\sum_{n=0}^{N-1}\sum_{m=0}^{n} B_n(\Lambda_A^+,t)(\Lambda_A^+)^m\Big\{\big(T_{nm}(P'-P_\infty)-(P-P_\infty)T_{nm}\big)_{F(A)} -$$
$$-i\,\mathcal{Y}\big(T_{nm}(P'-P_\infty)-(P-P_\infty)T_{nm}\big)_{O(A)}\cdot P' +$$
$$+i\,P\,\mathcal{Y}\big(T_{nm}(P'-P_\infty)-(P-P_\infty)T_{nm}\big)_{O(A)}\Big\}=0. \tag{2.19}$$

Thus, we have found the form of transformations of the potential $P(x,t)$, which correspond to the transformations of the scattering matrix of the form (2.4): these transformations are given by the relation (2.19) where $B_n(\lambda,t)$ is arbitrary entire functions.

The reason, for which we have confined ourselves to the transformations of the scattering matrix of the form (2.4), is clear now: for transformations of this type we can find, in an explicit form (i.e. in the form containing only P and P'), the corresponding transformations of the potential. It is remarkable that these "restricted" transformations (2.4) and (2.19) are quite wide and, as we shall see, contain all transformations typical for equations integrable by means of the problem (1.1) and these integrable equations themselves.

Transformations of the type (2.19) (in the case N = 2 and P_∞ = 0) have been considered by Calogero and Degasperis /23/ for the first time. Just these authors have turned attention to their significance.

At P_∞ = 0 the transformations (2.19) coincide, with an accuracy to the redefinition of the functions B_n, with the transformations constructed in Refs. /7-9/.

III. The general form of integrable equations; integrals of motion

Transformations (2.19) form, as is easy to see from (2.4), an infinite-dimensional Abelian group. This group contains the transformations of various types. Let us examine the one-parameter subgroup of this group, which is given by the matrix

$$B_n = e^{-i\int_t^{t'} ds\,\Omega_n(\lambda,s)} \tag{3.1}$$

where $\Omega_n(\lambda,t)$ are some entire functions and C = B. There is no difficulty in justifying that these transformations are time displacements: $S'(\lambda,t) = S(\lambda,t')$. The corresponding transformations (2.19) give, in the explicit form, the time evolution of the potential P: $P(x,t) \to P(x,t')$. Different evolutions laws correspond to different functions $\Omega_n(\lambda,t)$. An identical transformation is given by the functions $B_0 \equiv 1$, $B_1 = \ldots = B_{N-1} = 0$. Let us examine the infinitesimal displacement in time: $t \to t' = t+\varepsilon$ where $\varepsilon \to 0$. In this case,

$$P(x,t') = P(x,t) + \varepsilon\frac{\partial P}{\partial t},$$
$$B_n(\lambda,t) = \delta_{nc} - i\varepsilon\,\Omega_n(\lambda,t). \tag{3.2}$$

Substituting (3.2) into (2.19) and keeping the terms of first order in ε, we obtain

$$\frac{\partial P(x,t)}{\partial t} + i\sum_{n=0}^{N-1}\sum_{m=0}^{n}\Omega_n(L_A^+,t)(L_A^+)^m\Big\{[T_{nm},P-P_\infty]_{F(A)} + i\big[P,\mathcal{Y}[T_{nm},P]_{O(A)}\big]\Big\}=0 \tag{3.3}$$

where $L^+ \overset{def}{=} \Lambda^+(P'=P)$, i.e.

$$L^+ = i\frac{\partial}{\partial x} + [P(x),\cdot\,]_{F(A)} + i\big[P(x),\mathcal{Y}[P,\cdot\,]_{O(A)}\big]. \tag{3.4}$$

Correspondingly, for the scattering matrix from (2.4) we have

$$\frac{dS(\lambda,t)}{dt} = i\Big[\sum_{n=1}^{N-1}\Omega_n(\lambda,t)\bar{A}^n, S(\lambda,t)\Big]. \tag{3.5}$$

If one uses the equality

$$\sum_{m=0}^{n}(L_A^+)^m\Big\{[T_{nm},P-P_\infty]_{F(A)} + i\big[P,\mathcal{Y}[T_{nm},P]_{O(A)}\big]\Big\} = \sum_{m=0}^{n}(L_A^+)^m\,\big[(T_{nm})_{O(A)},P\big]$$

equations (3.3) may be readily written out in a more convenient form:

$$\frac{\partial P(x,t)}{\partial t}+i\sum_{n=1}^{N-1}\sum_{m=0}^{n}\Omega_n(L_A^+,t)(L_A^+)^m\left[(T_{nm})_{0(A)},P\right]=0 \qquad (3.6)$$

Thus, we have obtained the evolution differential equations as the infinitesimal form of transformations (2.19).

An insignificant modification of the constructions mentioned above makes it possible to show that the more general class of equations, namely, the equations of the form (3.6) with arbitrary functions $\Omega_n(\lambda,t)$, meromorphic over λ, is connected with the bundle (1.1).

A broader class of equations may be represented in the form (3.6), if P and Ω_n depend on several variables of time type. These are the equations

$$\sum_{i=1}^{p} f_i(L_A^+,t_1,\dots,t_p)\frac{\partial P(x,t_1,\dots,t_p)}{\partial t_i}+ i\sum_{n=1}^{N-1}\sum_{m=0}^{n}\Omega_n(L_A^+,t_1,\dots,t_p)(L_A^+)^m\left[(T_{nm})_{0(A)},P\right]=0 \qquad (3.7)$$

where $f_i(\lambda,t_1,\dots,t_p)\ (i=1,\dots,p)$ and $\Omega_n(\lambda,t_1,\dots,t_p)$ $(n=1,\dots,N-1)$ are arbitrary functions. For the case $N=2$, $P_\infty=0$ see Ref. /23/.

Evolution partial differential equations (3.6) are just the equations integrable by the IST method with the help of the bundle (1.1). Using the IST equations (Gelfand-Levitan-Marchenko equations), one can find, in principle, a broad class of exact solutions of equations of the form (3.6) (multi-soliton solutions) (see Refs. /1,2/).

Choosing concrete functions $\Omega_n(\lambda,t)$, there is no difficulty in obtaining a number of concrete examples of equations (3.6). At $P_\infty=0$ we return to the equations considered in Refs. /4,5,7-9/ (with an accuracy of the redefinition of the functions Ω_n). In a particular case when $\Omega_2=\Omega_3=\dots\Omega_{N-1}=0$, equations (3.6) is equivalent to the following one:

$$\frac{\partial P(x,t)}{\partial t}+\Omega_1(L_A^+,t)\frac{\partial P}{\partial x}=0\,. \qquad (3.8)$$

At N = 2 formula (3.8) gives the general form of equations integrable by means of (1.1). Moreover, if $(P_\infty)_{21}=(P_\infty)_{12}$ (and N = 2), we obtain the equations considered in the paper /24/ by another method.

Let us now turn our attention to the fact that at any functions $\Omega_n(\lambda,t)$ the quantity $S_{0}(\bar{A})$, by virtue of (3.5), is time independent:

$$\frac{d}{dt}S_0(\bar{A})=0\,. \qquad (3.9)$$

Thus the diagonal elements of the scattering matrix, at any λ, are the integrals of motion. Just as in the case $P_\infty=0$ (see /2,9/), one can extract a counting set of explicit and local integrals of motion from this continual set of inexplicit integrals of motion. Indeed, expanding $\ln S_{ii}(\lambda)$ in asymptotic series of λ^{-1},

$$\ln S_{ii}(\lambda)=\sum_{n=1}^{\infty}\lambda^{-n}C_i^{(n)}\qquad (i=1,\dots,N) \qquad (3.10)$$

we obtain the counting set of integrals of motion $C_i^{(n)}$ (i = 1,...,N; n = 1,2,3,...). These integrals of motion are of the form:

$$C_i^{(n)}=\int_{-\infty}^{+\infty}dx\,\left(\chi_i^{(n)}(x,t)-\chi_i^{(n)}(\infty,t)\right) \qquad (3.11)$$

where the quantities $\chi_i^{(n)}(x,t)$ are calculated from the standard recurrence relations similar to those which occur in the case $P_\infty=0$ /9/.

An important property of the integrals of motion is their universality: they are the integrals of motion for any equations of the form (3.6). Really, in their construction we have taken advantage of the time independence of $S_0(\bar{A})$ and the form of the bundle (1.1) rather than the form of the functions

$\Omega_n(\lambda,t)$. The universal character of the integrals of motion $C_i^{(n)}$ indicates that their existence is not associated with a concrete structure of equations (3.6), but only with the fact that these equations are of the form (3.6).

IV. Hamiltonian and Lagrangian structures of integrable equations; the classic r - matrix

Let us consider the equation of the form (3.6) with

$$\Omega_n(\lambda,t) = \sum_{m=0}^{\infty} \omega_{nm}(t)\lambda^m \tag{4.1}$$

where $\omega_{nm}(t)$ are arbitrary functions. In Appendix it is shown that the following relation

$$(L_A^+ - \lambda)\left[A, \Pi^{(n)}\right] = -\sum_{m=0}^{n} \lambda^m \left[(T_{nm})_{O(A)}, P(x,t)\right] \tag{4.2}$$

holds where

$$\left(\Pi^{(n)}(x,t,\lambda)\right)_{kp} \stackrel{def}{=\!=} \sum_{\ell=1}^{N} \left(\bar{A}^{-n}\right)_{\ell\ell} \frac{\left(\bar{\Phi}^{+(\ell\ell)}(x,t,\lambda)\right)_{kp}}{S_{\ell\ell}(\lambda)}.$$

Expanding the equality (4.2) in asymptotic series of λ^{-1}, we obtain

$$\left[A, \Pi^{(n)}_{n+1}\right] = -\sum_{\ell=0}^{n} (L_A^+)^\ell \left[(T_{n\ell})_{O(A)}, P\right] \tag{4.3}$$

and

$$L_A^+\left[A, \Pi_m^{(n)}\right] = \left[A, \Pi_{m+1}^{(n)}\right] \quad (m = n+1, n+2, \ldots) \tag{4.4}$$

where $\Pi^{(n)}(x,t,\lambda) = \sum_{m=0}^{\infty} \lambda^{n-m}\, \Pi_m^{(n)}(x,t)$.

Using (4.3) and (4.4), equation (3.4) with functions of the form (4.1) may be rewritten as follows:

$$\frac{\partial P(x,t)}{\partial t} = i\sum_{n=1}^{N-1}\sum_{\ell=0}^{\infty} \omega_{n\ell}(t)(L_A^+)^q (L_A^+)^{\ell-q}\left[A, \Pi_{n+1}^{(n)}\right]. \tag{4.5}$$

However, by virtue of (4.4),

$$(L_A^+)^{\ell-q}\left[A, \Pi^{(n)}_{n+1}\right] = \left[A, \Pi^{(n)}_{n+1+\ell-q}\right].$$

Since

$$\Pi^{(n)}_{n+1+\ell-q} = \frac{1}{(n+1+\ell-q)!}\frac{\partial^{n+1+\ell-q}}{\partial(\lambda^{-1})^{n+1+\ell-q}}\,\Pi^{(n)}(x,t,\lambda)\Big|_{\lambda=\infty}$$

equation (4.5) (and hence (3.6)) is equivalent to the following one:

$$\frac{\partial P}{\partial t} = (L_A^+)^q\left[A, i\sum_{n=1}^{N-1}\sum_{\ell=0}^{\infty}\omega_{n\ell}(t)\frac{1}{(n+1+\ell-q)!}\frac{\partial^{n+1+\ell-q}\Pi^{(n)}(x,t,\lambda)}{\partial(\lambda^{-1})^{n+1+\ell-q}}\Big|_{\lambda=\infty}\right] \tag{4.6}$$

where q is arbitrary integer.

Let us use now the relation (2.3). It follows from it that

$$\delta S_{in} = -i\int_{-\infty}^{+\infty} dx \sum_{k,\ell=1}^{N} \delta P_{k\ell}(x,t)\left(\bar{\Phi}^{+(in)}(x,t,\lambda)\right)_{\ell k} \quad (i, n = 1,\ldots,N)$$

where δS and δP are arbitrary variations consistent with (1.1). Hence

$$\left(\bar{\Phi}^{+(nm)}(x,t,\lambda)\right)_{\ell k} = i\frac{\delta}{\delta P_{k\ell}(x,t)} S_{nm}(\lambda,t) \tag{4.7}$$

$$(k, \ell, m, n = 1,\ldots,N)$$

where $\frac{\delta}{\delta P}$ is a variational derivative. For the quantity $\Pi^{(n)}(x,t,\lambda)$ we have

$$\Pi^{(n)}(x,t,\lambda) = i\frac{\delta}{\delta P^T(x,t)}\, tr\left(\bar{A}^{-n}\ln S_{O(\bar{A})}(\lambda)\right) \tag{4.8}$$

where P^T stands for the transposed matrix P.

By virtue of (4.8), equation (4.6) may be represented as follows:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{1}{2}(L_A^+)^q\left[A, \frac{\delta \mathcal{H}_{-q}}{\delta P^T}\right] \tag{4.9}$$

where

$$\mathcal{H}_{-q} = 2\sum_{n=1}^{N-1}\sum_{\ell=0}^{\infty} \omega_{n\ell}(t)\frac{1}{(n+1+\ell-q)!}\frac{\partial^{n+1+\ell-q}}{\partial(\lambda^{-1})^{n+1+\ell-q}} tr(\bar{A}^{n} \ln S_{0(\lambda)})\Big|_{\lambda=\infty}. \tag{4.10}$$

It is easy to see that equation (4.9) may be written out in the Hamiltonian form:

$$\frac{\partial P(x,t)}{\partial t} = \{ P(x,t), \mathcal{H}_{-q} \}_q$$

with respect to the infinite set of Hamiltonians $\mathcal{H}_{-q}$ (4.9) and the Poisson brackets:

$$\{\mathcal{F}, \mathcal{H}\}_q \stackrel{def}{=} -\frac{1}{2}\int_{-\infty}^{+\infty} dy\, tr\left(\frac{\delta \mathcal{F}}{\delta P^T(y)}(L_A^+)^q\left[A, \frac{\delta \mathcal{H}}{\delta P^T}\right]\right) \tag{4.11}$$

where q is an arbitrary integer. The fact that the bracket (4.11) is really the Poisson brackets (skew-symmetry, Jacoby identity) is justified by direct calculation.

Thus we have shown that equations (3.6) are Hamiltonian ones with respect to the infinite set of Hamiltonian structures. The existence of the infinite set of Poisson brackets of the type (4.11) was pointed out in Ref. /25/ for the first time. The hierarchy of Hamiltonian structures for equations of the type (3.6) with N = 2, P_∞ = 0 was analysed in Ref. /26/. The general theory of the structures of such a type was discussed in Refs. /27,28/.

The set of closed symplectic 2 - forms corresponding to the brackets (4.11) looks as follows:

$$\omega^{\ell}(\delta_1 P, \delta_2 P) = \frac{1}{2}\int_{-\infty}^{+\infty} dx\, tr\left(\delta_2 P_A(x)(L_A^+)^{\ell}\delta_1 P - \delta_1 P_A(x)(L_A^+)^{\ell}\delta_2 P\right) \tag{4.12}$$

where ℓ is an arbitrary integer. For the case N = 2, P_∞ = 0 and $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ see Ref. /26/. A group-theoretical interpretation of the structures of the type (4.12) is given in /28/.

Note that the Poisson brackets (4.11) and the forms (4.12) are the same for all equations of the (3.6). Therefore, the

phase space of the dynamical systems, which are described by equations (3.6), has an universal symplectic structure in the general position.

At N = 2, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ the simplest Poisson bracket $\{\ ,\ \}_0$ coincides with the well known bracket /1,2/

$$\{\mathcal{F}, \mathcal{H}\}_0 = \int_{-\infty}^{+\infty} dy\left(\frac{\delta \mathcal{F}}{\delta r(y,t)}\frac{\delta \mathcal{H}}{\delta q(y,t)} - \frac{\delta \mathcal{F}}{\delta q(y,t)}\frac{\delta \mathcal{H}}{\delta r(y,t)}\right). \tag{4.13}$$

For the case N = 2 and P_∞ = 0 the Hamiltonian structure of equations of the form (3.6) was considered in Ref. /29/ in considerable details.

It is noteworth that, by virtue of (3.10), the Hamiltonians (4.10) are superpositions of the integrals of motion $C_i^{(n)}$ and, therefore, they are local and explicit functionals of the potential $P(x,t)$. One can also show that all integrals of motion $C_i^{(n)}$ $(i = 1,\dots, N;\ n=1,2,\dots)$ are in involution, i.e. that $\{C_i^{(n)}, C_k^{(m)}\}_q = 0$ for all i, k, m, n and q.

Let us touch upon the Lagrangian structure of equations (3.6). Since these equations are Hamiltonian and the Poisson bracket $\{\ ,\ \}_0$ is canonical (the quantities $P(x,t)$ and $2(P^T(x,t))_A$ form a pair of canonically conjugated dynamical variables), there is no difficulty in constructing the Lagrangian $\mathcal{L}$ over Hamiltonian $\mathcal{H}$: it is quite enough to use the usual Legendre transformation (see, e.g. /30/). Hence, for equations of the form (3.6) the Lagrangian $\mathcal{L}$ is equal to

$$\mathcal{L} = -2\, tr\left(P\frac{\partial P_A}{\partial t}\right) - \tilde{\mathcal{H}} \tag{4.14}$$

where $\tilde{\mathcal{H}}$ is the Hamiltonian density $\left(\mathcal{H} = \int_{-\infty}^{+\infty} dx\, \tilde{\mathcal{H}}(x)\right)$. Indeed, the Euler equation

$$\frac{\delta \mathcal{L}}{\delta P} = -2\frac{\partial}{\partial t}(P_A)^T - \frac{\delta \tilde{\mathcal{H}}}{\delta P} = 0$$

is equivalent, as is easy to see, to equation (4.9) with q = 0.

The general theorem on the Lagrangian nature of equations integrable by the rational bundle (in another gauge), was proved in Ref. /31/.

In conclusion of this section, we are going to calculate the Poisson bracket between the elements of the scattering matrix. The knowledge of these Poisson brackets is necessary to construct variables of the action-angle type.

So, let us consider the Poisson bracket $\{S_{i_1k_1}(\lambda_1), S_{i_2k_2}(\lambda_2)\}_0$. By virtue of (4.11) and (4.7), we have

$$\{S_{i_1k_1}(\lambda_1), S_{i_2k_2}(\lambda_2)\}_0 = \frac{1}{2}\int_{-\infty}^{+\infty} dx\, tz\left(\overset{-+}{\Phi}{}^{(i_1k_1)}(x,\lambda_1)\left[A, \overset{-+}{\Phi}{}^{(i_2k_2)}(x,\lambda_2)\right]\right). \tag{4.15}$$

Using the equality $\lambda\left[A, \overset{-+}{\Phi}\right] = -i\frac{\partial \overset{-+}{\Phi}}{\partial x} - \left[P, \overset{-+}{\Phi}\right]$ from (4.15), we find

$$(\lambda_1-\lambda_2)\{S_{i_1k_1}(\lambda_1), S_{i_2k_2}(\lambda_2)\}_0 = \lim_{z\to\infty}\frac{i}{2}\, tz\left(\overset{-+}{\Phi}{}^{(i_1k_1)}(x,\lambda_1)\,\overset{-+}{\Phi}{}^{(i_2k_2)}(x,\lambda_2)\right)\Big|_{x=-z}^{x=+z}. \tag{4.16}$$

Since $\left(\overset{-+}{\Phi}{}^{(i_1k_1)}(\lambda)\right)_{nm} \overset{\mathrm{def}}{=} \left(F^{+}(\lambda)\right)_{nk_1}\left(F^{-}(\lambda)\right)^{-1}_{i_1m}$, we obtain

$$(\lambda_1-\lambda_2)\{S_{i_1k_1}(\lambda_1), S_{i_2k_2}(\lambda_2)\}_0 =$$

$$= \lim_{z\to\infty}\frac{i}{2}\left(\left(F^{-}(x,\lambda_2)\right)^{-1}F^{+}(x,\lambda_1)\right)_{i_2k_1}\left(\left(F^{-}(x,\lambda_1)\right)^{-1}F^{+}(x,\lambda_2)\right)_{i_1k_2}\Big|_{x=-z}^{x=+z}. \tag{4.17}$$

Taking into account the asymptotic properties of F^{+} and F^{-}, we have

$$\{S_{i_1k_1}(\lambda_1), S_{i_2k_2}(\lambda_2)\}_0 =$$

$$= S_{i_1m}(\lambda_1)S_{i_2n}(\lambda_2)\, r_{mnk_1k_2}(\lambda_1,\lambda_2) - r_{i_1i_2mn}(\lambda_1,\lambda_2)\, S_{mk_1}(\lambda_1)S_{nk_2}(\lambda_2) \tag{4.18}$$

where

$$r_{mnk_1k_2}(\lambda_1,\lambda_2) = \frac{i}{2}\lim_{z\to\infty}\frac{1}{\lambda_1-\lambda_2}\left(E^{-1}(z,\lambda_1)E(z,\lambda_2)\right)_{mk_2}\left(E^{-1}(z,\lambda_2)E(z,\lambda_1)\right)_{nk_1}. \tag{4.19}$$

The matrix $r_{i_1i_2nm}(\lambda_1,\lambda_2)$ is none other than the classic r - matrix (see, e.g. /32,33/). The method, by which we have derived (4.18), is close to that used in Refs. /34,35/.

Taking into account (2.1), the r - matrix may be rewritten as follows:

$$r_{mnk_1k_2}(\lambda_1,\lambda_2) = \frac{i}{2}\left(D^{-1}(\lambda_1)D(\lambda_2)\right)_{mk_2}\left(D^{-1}(\lambda_2)D(\lambda_1)\right)_{nk_1}\cdot$$

$$\cdot\lim_{z\to\infty}\frac{\exp i(-\mu_{1m}+\mu_{2k_2}-\mu_{2n}+\mu_{1k_1})z}{(\lambda_1-\lambda_2)} \tag{4.20}$$

$$(m, n, k_1, k_2 = 1,\dots,N)$$

To find the explicit form of the r - matrix, it is necessary to know the matrix $D(\lambda)$ and the eigenvalues of the matrix $\lambda A + P_\infty$.

For the sake of simplicity, we shall consider the case of the diagonal matrix A ($A_{ik} = a_i\delta_{ik}$, $a_i \neq a_k$, and all a_i are real) and $P_\infty = 0$. Taking into account the known equality $V.p. \lim_{z\to\infty}\frac{\exp(ia\xi z)}{\xi} = i\pi\,\mathrm{sgn}\,a\,\delta(\xi)$, we obtain

$$r_{mnk_1k_2}(\lambda_1,\lambda_2) = r_{mnk_1k_2}(\lambda_1-\lambda_2) =$$

$$= \frac{i}{2}\delta_{mk_2}\delta_{nk_1}\left(\frac{\delta_{k_1k_2}}{\lambda_1-\lambda_2} + i\pi\,\mathrm{sgn}(a_{k_1}-a_{k_2})\,\delta(\lambda_1-\lambda_2)\right) \tag{4.21}$$

where $\mathrm{sgn}(\xi) = \begin{cases}1, & \xi>0\\ -1, & \xi<0\end{cases}$ and $\mathrm{sgn}(0) = 0$. Note that $\{S_{i_1i_1}(\lambda_1), S_{i_2i_2}(\lambda_2)\}_0 = 0$ $(i_1, i_2 = 1,\dots,N)$. It follows immediately from this that the integrals of motion $C_i^{(n)}$ are in involution. Indeed, by virtue of (3.10), $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\lambda_1)^{-m}(\lambda_2)^{-n}\{C_{i_1}^{(n)}, C_{i_2}^{(m)}\}_0 = 0$. Hence, $\{C_{i_1}^{(n)}, C_{i_2}^{(m)}\}_0 = 0$.

The Poisson brackets $\{ S_{i_1 k_1}(\lambda_1), S_{i_2 k_2}(\lambda_2) \}_q$ for any q will be calculated elsewhere.

V. Transformation properties of integrable equations

The group-theoretical properties of equations of the form (3.6) are completely analogous to the corresponding ones of integrable equations with $P_\infty = 0$ /36,37/. In view of this, we shall dwell upon them very briefly.

The infinite-dimensional group transformations (2.19) contains the infinite-dimensional Abelian group of symmetry of equations (3.6) as a subgroup. In the infinitesimal form the symmetry transformations are the following ($P \to P' = P + \delta P$):

$$\delta P = \sum_{n=1}^{N-1} \sum_{m=0}^{n} f_n(L_A^+)(L_A^+)^m \left[(T_{nm})_{D(A)}, P \right] \qquad (5.1)$$

where $f_n(\lambda)$ are arbitrary entire functions.

It should be mentioned that the group of symmetry (5.1) is universal: any from the equations of the form (3.6) is invariant under transformations (5.1). The groups of symmetry and the integrals of motion $C_i^{(n)}$ posses the universality of the same nature.

Transformations (2.19) with functions B_n, which are time independent, are auto-Backlund transformations: they transform the solutions of a certain equation of the form (3.6) into the solutions of the same equation. Just like to the case $P_\infty = 0$, discrete auto-Backlund transformations may be represented in the form of combinations of elementary Backlund transformations $B^{(\alpha)}_{\lambda_0^{(\alpha)}}$ $(P \to P')$, i.e. transformations (2.19) with functions B_n, such that $B_\alpha = \lambda - \lambda_0^{(\alpha)}$, $B_1 = \ldots B_{\alpha-1} = B_{\alpha+1} = \ldots B_N \equiv 1$. The known soliton Backlund transformations are particular cases of discrete Bäcklund ones.

Transformations (2.19) with time dependent functions B_n transform the solutions of a certain equation of the form (3.6) into the solutions of another equation of the form (3.6). Transformations of such a type are referred to as generalized Back-

lund transformations (see /23/). Generalized Backlund transformations act in a transitive manner on the whole set of equations of the type (3.6).

Thus transformations (2.19) is a condenced expression of the transformation properties of equations integrable by means of the bundle (1.1).

VI. The reduction problem in the AKNS-method

The number of fields in equations (3.6) (i.e. the number of components of the potential $P(x,t)$) grows fastly with increasing N (as $N^2 - N$). In this connection, the reduction problem for the general equations arises, i.e. the problem of decreasing the number of independent fields. To do this, some restrictions are imposed on the potential $P(x,t)$, these restrictions being consistent with equation (3.6). In this case, the reduction is said to be made in the general system (3.6) (for the formulation of the reduction problem see, e.g. /38,39, 1,20/. Not long ago Mikhailov /14,15/ has made a significant step in this direction (also, see /16,17/).

Within the framework of the AKNS-method the reduction problem is divided into two problems. The first one is to construct a non-trivial group of reduction for the bundle (1.1) and the second is to find the class of functions $\Omega_n(\lambda,t)$, for which equations (3.6) allow this reduction.

Let us recall briefly the approach to the reduction problem, which has been proposed by Mikhailov /14,15/. Let

$$\frac{\partial \Psi}{\partial x} = U(x;\lambda)\, \Psi(x,\lambda) \qquad (6.1)$$

be a linear spectral problem. The existence of the non-trivial reduction in the system (6.1) is connected with the form-invariance of the problem (6.1) under the transformations:

$$\Psi(x,\lambda) \to \Psi'(x,\lambda') = G(x,\lambda)\, \Psi(x,\lambda), \qquad (6.2)$$

$$\lambda \to \lambda' = g(\lambda), \qquad (6.3)$$

$$X \to X' = f(X) \qquad (6.4)$$

where (6.3) is a certain transformation in the complex plane of the variable λ and (6.4) is a transformation of the variable X.

The invariance condition under the transformations (6.2)-(6.4) means that $\Psi'(X', \lambda')$ is also a solution of the problem (1.1) and that

$$U(f(x); g(\lambda)) = G(x,\lambda)\, U(x;\lambda)\, G^{-1}(x,\lambda) + \frac{\partial G(x,\lambda)}{\partial x} G^{-1}(x,\lambda) \qquad (6.5)$$

The condition (6.5) of form-invariance of the potential $U(x;\lambda)$ under transformations (6.2)-(6.4) is a system of equations on $U(x;\lambda)$. Solving equation (6.5), we obtain the smaller number of independent variables.

In our case (for the bundle (1.1)) the condition (6.5) is equivalent to the following one (for simplicity we assume that $X' = X$, $\frac{\partial G}{\partial x} = 0$):

$$g(\lambda) A + P(x,t) = G(\lambda)(\lambda A + P(x,t))\, G^{-1}(\lambda) \qquad (6.6)$$

Thus, to enumerate reductions in the general system (3.6), the non-trivial solutions of the system (6.6) should be found*. As a result, we obtain the solution of the first part of the reduction problem.

Let us turn now to the problem of enumerating the functions Ω_n, for which equations (3.6) allow the reduction (6.6).

Let $\Psi(x,\lambda)$ be any fundamental matrix-solution of the system (1.1). Since $\Psi'(x, g(\lambda))$ is also a matrix-solution of the system (1.1), there exists such non-degenerated matrix $T(\lambda)$ that

$$\Psi'(x, g(\lambda)) = \Psi(x, g(\lambda))\, T(\lambda) \qquad (6.7)$$

Combining (6.7) and (6.2), we obtain

* Note that reductions of the type (6.6) do not exhaust all possible reductions of the equations (3.6).

$$G(\lambda)\Psi(x,\lambda) = \Psi(x, g(\lambda))\, T(\lambda). \qquad (6.8)$$

The condition (6.8) is satisfied both for the solutions F^+ and F^-. Since $F^+(\lambda) = F^-(\lambda) S(\lambda)$ and $F^+(g(\lambda)) = F^-(g(\lambda)) S(g(\lambda))$ then, by virtue of (6.8), the scattering matrix $S(\lambda, t)$ satisfies the following condition

$$S(g(\lambda), t) = T(\lambda)\, S(\lambda, t)\, T^{-1}(\lambda). \qquad (6.9)$$

Thereby, under reduction (6.6), for the elements of the scattering matrix the relations (6.9) hold, which decrease the number of independent elements of S.

In section III it is shown that if the evolution of the potential $P(x,t)$ is given by equation (3.6), then the evolution of the scattering matrix $S(\lambda, t)$ is given by equation (3.5). Therefore, along side with the consistence of (6.6) with (3.6) we should require simultaneously the consistence of (6.9) with equation (3.5). As a result, we obtain the following relation

$$Y(g(\lambda), t) = T(\lambda)\, Y(\lambda, t)\, T^{-1}(\lambda) \qquad (6.10)$$

where $Y(\lambda, t) \overset{def}{=} \sum_{n=1}^{N-1} \Omega_n(\lambda, t) \bar{A}^n$. Solving the system (6.10) with respect to Ω_n, we find the form of functions Ω_n at which equations (3.6) allow the reduction (6.6). Indeed, at such functions $\Omega_n(\lambda, t)$ the relation (6.9) is consistent with equation (3.6) and, hence the relation (6.6) is consistent with equation (3.6). Using the explicit form of operator L_A^+, one can directly justify that for functions Ω_n, which satisfy (6.10), the condition (6.6) is consistent with equation (3.6).

Thus, we have the following procedure for an analysis of the reductions of the general equations (3.6):

1) First of all, we find the reduction of the linear bundle (1.1), i.e. solving equation (6.6), we find $g(\lambda)$, the matrix $G(\lambda)$ and the form of matrices A and $P(x,t)$;

2) Then, we calculate the matrix $T(\lambda)$ according to the

formula*

$$T(\lambda) = E^{-1}\big(g(\lambda)\big)\, G(\lambda) E(\lambda), \qquad (6.11)$$

3) From the relation (6.10) we find the form of functions Ω_n for which equations (3.6) allow this reduction.

It should be emphasized that in the general case the matrix $T(\lambda)$ does not coincide with the matrix $G(\lambda)$: they are related by formula (6.11). The form of the matrix $T(\lambda)$, with a given matrix $G(\lambda)$, depends on a choice of the asymptotic of the linear problem (1.1). Note that in Refs. /14,15/ the case T = G was analysed.

The method considered above is effective for any reductions which are generated by linear constraints on the elements of the potential $P(x,t)$. It was suggested in Ref. /40/ in which some linear reductions were considered. A similar method for an analysis of reductions for the other spectral problem ($\frac{\partial^2\psi}{\partial x^2} + q(x)\psi = \lambda^2\psi$) has been examined in the papers by Calogero and Degasperis /16,17/.

In considering the reduction problem, the following important problem arises. As has been shown (section IV), equations (3.6) are Hamiltonian ones in the general position, meanwhile under reductions we have dynamical systems with constraints. There exists the well known Dirac method /41/ for an analysis of such systems, in which one goes without the solution of the constraints. Another, more direct, method is to solve the constraints between dynamical variables, to introduce new, independent dynamical variables, to rewrite equations in terms of independent dynamical variables and to study the Hamiltonian structure of these equations. We shall follow this direct method. Within the framework of the AKNS method one can show that under reductions the equations (3.6) are also Hamiltonian and calculate the set of the corresponding Hamiltonian structures. At N = 2, $P_\infty = 0$ the Hamiltonian structure of the reduced equations was studied in a remarkable paper /29/.

* The relation (6.11) is obtained from (6.8) at $|x| \to \infty$.

VII. Z_N - reduction

In this section (and in the nest ones) we shall consider some concrete reductions of equations (3.6) as the examples.

Reductions of the Z_N - type are connected with the group Z_N of transformations $\lambda \to \lambda' = q\lambda$ ($q = \exp\frac{2\pi i}{N}$, $q^N = 1$) of the complex plane of the variable λ . These reductions was analysed in Refs. /14,15/ for the first time. Equations (6.6) have a large number of solutions. We shall consider here the case of the diagonal matrix A and $P_\infty = 0$, $\frac{\partial G}{\partial \lambda} = 0$. From (6.6) we have

$$qAG = GA, \quad G^{-1}PG = P. \qquad (7.1)$$

Hence*

$$A = \begin{pmatrix} 1 & & & 0 \\ & q & & \\ & & \ddots & \\ 0 & & & q^{N-1} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad (7.2)$$

$$P = \begin{pmatrix} 0 & Q_{N-2} & \dots & Q_2 & Q_1 & Q_0 \\ Q_0 & 0 & Q_{N-2} & \dots & Q_2 & Q_1 \\ Q_1 & Q_0 & 0 & \dots & \dots & Q_2 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & Q_{N-2} \\ Q_{N-2} & Q_{N-3} & \dots & Q_1 & Q_0 & 0 \end{pmatrix}. \qquad (7.3)$$

Note that $A^N = 1$, $G^N = 1$.

Since $E = \exp i\lambda Ax$ from (6.11) we obtain T = G. Taking into account the relation $GAG^{-1} = qA$, from equation (6.10) we

* The solution of (7.1)-(7.3) was found in Ref. /14/ for the first time.

find that the functions Ω_n should satisfy the conditions $\Omega_n(q\lambda, t) = \Omega_n(\lambda, t)$ $(n=1,\dots,N)$. These conditions are satisfied by any functions λ^N, i.e. $\Omega_n = \Omega_n(\lambda^N, t)$.

Thus equations (3.6) allow the reduction (7.1)-(7.3) ((we shall refer it to as a pure Z_N-reduction), if $\Omega_n = \Omega_n(\lambda^N, t)$ $(n=1,\dots,N)$.

Let us rewrite now equation (3.6) under the pure Z_N-reduction in terms of N-1 independent variables $Q_0, Q_1, \dots, Q_{N-2}$. The constraint GP=PG are solved as follows. We introduce the matrix Q such that all its non-zero elements are independent. For example,

$$Q = \begin{pmatrix} 0 & Q_{N-2} & \dots & Q_2 & Q_1 & Q_0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \qquad (7.4)$$

Then, the potential P of the form (7.3) may be represented as follows:

$$P = \sum_{m=1}^{N} G^{-m} Q G^{m}. \qquad (7.5)$$

Let us introduce the projection operation Δ: $(\varphi_\Delta)_{ik} \stackrel{def}{=} \delta_{i1}\varphi_{1k}$. In particular, $P_\Delta = Q$.

One can show that the relations

$$L_{(n)} \overset{++}{\varphi}{}^{F(A)}_{(n)\Delta}(\lambda) = \lambda^N \overset{++}{\varphi}{}^{F(A)}_{(n)\Delta}(\lambda), \qquad (7.6)$$

$$\mathcal{L}_{(n)} \overset{++}{\varphi}{}^{F(A)}_{(0)}(\lambda) = \lambda^n \overset{++}{\varphi}{}^{F(A)}_{(n)\Delta}(\lambda) \quad (n=1,2,\dots,N)$$

hold /42/ where $\varphi_{(n)} \stackrel{def}{=} \sum_{m=1}^{N} q^{nm} G^m \varphi_{F(A)} G^{-m}$ and

$$L_{(n)}\varphi \stackrel{def}{=} \sum_{m=1}^{N} q^{-nm} \left((L_A)^N G^{-m} \varphi G^{m}\right)_\Delta$$

and

$$\mathcal{L}_{(n)}\varphi \stackrel{def}{=} \sum_{m=1}^{N} \left((L_A)^n G^{-m}\varphi G^{m}\right)_\Delta$$

and the operator $L \stackrel{def}{=} \Lambda(P'=P)$ where P is of the form (7.5).

Separating independent variables in all the relations and using (7.6), one can reduce equations (3.6) to the following form /42/:

$$\frac{\partial Q}{\partial t} - i\sum_{n=1}^{N-1}\Omega_n(L^+_{(0)}, t)\,\mathcal{L}^+_{(n)}[A^n, Q] = 0 \qquad (7.7)$$

where $L^+_{(n)}$ and $\mathcal{L}^+_{(n)}$ are the operators adjoint to $L_{(n)}$ and $\mathcal{L}_n$, i.e.

$$L^+_{(n)}\varphi = \sum_{m=1}^{N} q^{-nm}\left(G^m\left((L^+_A)^N\varphi\right)G^{-m}\right)_\Delta,$$

$$\mathcal{L}^+_{(n)}\varphi = \sum_{m=1}^{N} \left(G^m\left((L^+_A)^N\varphi\right)G^{-m}\right)_\Delta. \qquad (7.8)$$

Equations (7.7) are a form of equations (3.6) under a pure Z_N-reduction, which contains only independent variables $Q_0, Q_1, \dots, Q_{N-2}$.

Equations (7.7) are Hamiltonian ones. Indeed, one can write out them as follows: /42/

$$\frac{\partial Q}{\partial t} = (L^+_{(0)})^q\, \mathcal{D}_Q \frac{\delta \mathcal{H}_{-q}}{\delta Q^T} \qquad (7.9)$$

where q is an arbitrary integer, $\mathcal{H}_{-q}$ are some functionals and

$$\mathcal{D}_Q\varphi \stackrel{def}{=} \sum_{m=1}^{N} G^m(L^+\varphi)G^{-m}.$$

It is clear that equations (7.9) are representable in the Hamiltonian form:

$$\frac{\partial Q(x,t)}{\partial t} = \left\{ Q(x,t),\ \mathcal{H}_{-q} \right\}_q$$

with respect to the infinite set of Poisson brackets $\{\mathcal{F}, \mathcal{H}\}_q$:

$$\{\mathcal{F}, \mathcal{H}\}_q \overset{def}{=} \int_{-\infty}^{+\infty} dy \, tr \left(\frac{\delta \mathcal{F}}{\delta Q^T(y)} \left(L^+_{(0)} \right)^q \mathcal{D}_Q \frac{\delta \mathcal{H}}{\delta Q^T} \right) \qquad (7.10)$$

where q is an arbitrary integer. The proof of the fact that the brackets (7.10) are Poisson brackets is cumbersome.

It is obvious that the set of brackets (7.10) is obtained as a result of the Z_n reduction of the set of brackets (4.11). Note that under the Z_n reduction $\{\mathcal{F}, \mathcal{H}\}_q |_{PG=GP} = 0$ for any functionals $\mathcal{F}$ and $\mathcal{H}$ if $q \neq kN-1$ (k = 0, ±1, ±2, ...).

The behaviour of the set of symplectic forms (4.12) is determined by the following

<u>Theorem</u> /42,43/. Under a pure Z_N - reduction (7.1)-(7.3) one have

$$\omega^{(\ell)} (\delta_1 P, \delta_2 P) \Big|_{PG=GP} = 0$$

if $\ell \neq kN-1$ $(k = 0, \pm 1, \pm 2, \ldots)$ and

$$\omega^{(kN-1)} (\delta_1 P, \delta_2 P) \Big|_{PG=GP} \overset{def}{=} \omega^k_Q (\delta_1 Q, \delta_2 Q) =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} dx \, tr \left(\delta_2 Q(x) \mathcal{D}_Q^{-1} (L^+_{(0)})^k \delta_1 Q - \delta_1 Q(x) \mathcal{D}_Q^{-1} (L^+_{(0)})^k \delta_2 Q \right) \qquad (7.11)$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

There is no difficulty in seeing that two-forms (7.11) correspond to the Poisson brackets (7.10) and vice versa.

Note that equations (7.7), as a particular case, contain the equations of the two-dimensional Toda lattice, considered in /14/ ($\Omega_1 = \ldots = \Omega_{N-2}$, $\Omega_{N-1} = \lambda^N$).

The matrix pure Z_N -reduction may be considered both in a gauge of the type (7.2) and in the so-called R - gauge in

a similar way. (For details see Ref. /42/).

At N = 2 the Z_2 - reduction means that $P_{12} = P_{21}$ and equations (7.7) are equivalent to the equations considered in Ref. /29/.

VIII. $\widetilde{Z}_N$ - reduction; connection with the Gelfand-Dikij spectral problem

We are going to consider another solution of equation (6.6) with $q = \exp \frac{2\pi i}{N}$. Let $P_\infty \neq 0$ and $\frac{\partial G}{\partial \lambda} \neq 0$. One of the solutions of the system (6.6) is of the form

$$A = \begin{pmatrix} 1 & & 0 \\ & q & \\ & & \ddots \\ 0 & & & q^{N-1} \end{pmatrix}, \quad G(\lambda) = \begin{pmatrix} 1 & \lambda(q-1) & 0 & \ldots & 0 \\ 0 & 1 & \lambda(q-1) & \ldots & 0 \\ \vdots & & \ddots & \ddots & \\ & & & 1 & \lambda(q-1) \\ 0 & \ldots & & 0 & 1 \end{pmatrix}, \qquad (8.1)$$

$$P_{\widetilde{Z}_N} = \begin{pmatrix} 0 & Q_{N-2} & Q_{N-3} & \ldots & Q_2 & Q_1 & Q_0 \\ 1 & 0 & Q_{N-2} & & \ddots & Q_2 & Q_1 \\ 0 & 1+q & 0 & Q_{N-2} & \ddots & & Q_2 \\ 0 & 0 & 1+q+q^2 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & 0 & Q_{N-2} \\ 0 & 0 & \ldots & & 0 & 1+q+\ldots+q^{N-2} & 0 \end{pmatrix}. \qquad (8.2)$$

We refer to the reduction (8.1)-(8.2) as the $\widetilde{Z}_N$ reduction. For this reduction $G(q^{N-1}\lambda) G(q^{N-2}\lambda) \ldots G(\lambda) = 1$ and $\left(A + \frac{P_\infty}{\lambda}\right)^N = 1$, while for the pure Z_N reduction $G^N = A^N = 1$. Calculating the asymptotic $E(\lambda)$ from (6.11) we find

$$T(\lambda)=\begin{pmatrix} 0 & \lambda(q-1) & 0 & \dots & 0 \\ 0 & 0 & \lambda(q-1) & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & \lambda(q-1) \\ (\lambda(q-1))^{1-N} & 0 & & \dots & 0 \end{pmatrix} \tag{8.3}$$

Note that $T(q^{N-1}\lambda)\dots T(q\lambda)T(\lambda)=1$. Hence the matrices $G(\lambda)$ and $T(\lambda)$ are different representations of the same $\tilde{Z}_N$ -reduction group.

Using (8.3), from equations (6.10) we obtain $\Omega_n(q\lambda,t)=\Omega_n(\lambda,t)$ $(n=1,\dots,N-1)$. Hence equations (3.6) admit the reduction (8.1)-(8.2), if $\Omega_n=\Omega_n(\lambda^N,t)$ $(n=1,\dots,N-1)$.

Just as in the case of the pure Z_N reduction, equations (3.6) may be rewritten in the form which contains only the independent dynamical variables $Q_0, Q_1,\dots, Q_{N-2}$ and show that they are Hamiltonian with respect to the infinite set Hamiltonian structures.

Here we would like to attract attention to the connection of the linear bundle (1.1) under the $\tilde{Z}_N$ - reduction with the other important type of spectral problems.

Lemma /42,43/. Under the $\tilde{Z}_N$ -reduction (8.1)-(8.2) the spectral problem (1.1) is equivalent to the Gelfand-Dikij spectral problem: $\sum_{K=0}^{N} V_K(x)\left(-i\frac{\partial}{\partial x}\right)^K \chi=\lambda^N\chi$ with $V_n=1$, $V_{N-1}=0$. Coefficients $V_0, V_1,\dots, V_{N-2}$ are connected with $Q_0, Q_1,\dots, Q_{N-2}$ by the following relations:

$$\sum_{K=0}^{N} V_K(x)\left(P^{(K)}(x)\right)_{N\ell}=0 \quad (\ell=1,\dots,N) \tag{8.4}$$

where $P^{(K+1)}=-i\frac{\partial}{\partial x}P^{(K)}+P^K\cdot P_{\tilde{Z}_N}$, $P_0\stackrel{def}{=}1$ (k = 1,..., N - 1).

The Lemma is proved by direct calculation for $\chi=\Psi_N$ where $\Psi=(\Psi_1,\dots,\Psi_N)^T$. Relations (8.4) are readily solved with respect to Q_K and V_K. For N = 2 (q = -1) we have the well known result $Q_0=-V_0$. At N = 3 $(q=\exp\frac{2\pi i}{3})$

$$V_0=-i(1+q)\left(\frac{\partial Q_1}{\partial x}-iQ_0\right),\quad V_1=-(2+q)Q_1$$

or

$$Q_0=-\frac{1}{1+q}V_0+\frac{i}{2+q}\frac{\partial V_1}{\partial x},\quad Q_1=\frac{-1}{2+q}V_1.$$

At N = 4 $(q=\exp\frac{2\pi i}{4})$ we have

$$V_2=-(3+2q+q^2)Q_2,$$

$$V_1=i(3+3q+2q^2)\frac{\partial Q_2}{\partial x}-(1+q)(2+q+q^2)Q_1,$$

$$V_0=(1+q+q^2)\left\{Q_2^2+\frac{\partial^2 Q_2}{\partial x^2}+i(1+q)\frac{\partial Q_1}{\partial x}-(1+q)Q_0\right\}.$$

For arbitrary N, in particular, $V_{N-2}=$

$$=-\left(\sum_{K=1}^{N-1}\sum_{i=1}^{K-1}q^{i-1}\right)Q_{N-2}$$

It follows from this Lemma that equations integrable by the Gelfand-Dikij spectral problem /44/ are equivalent to equations (3.6) under the Z_N reduction.

Equations which are associated with the Gelfand-Dikij spectral problem are of the form /44/

$$\frac{\partial V_r}{\partial t}=\sum_{S=0}^{N-2}\ell_{rS}\frac{\delta\mathcal{H}}{\delta V_S}\quad(r=0,\dots,N-2). \tag{8.5}$$

Also they are Hamiltonian with respect to the Poisson bracket (The Gelfand-Dikij bracket) /44/:

$$\{\mathcal{F},\mathcal{H}\}_{G.D.} = \sum_{r,s=0}^{N-2} \int_{-\infty}^{+\infty} dx \, \frac{\delta \mathcal{F}}{\delta V_r} \, l_{rs} \, \frac{\delta \mathcal{H}}{\delta V_s} . \tag{8.6}$$

The relation between the Hamiltonian structures (8.6) and the set of Hamiltonian structures (4.12) under the $\widetilde{Z}_N$ reduction is given by the following

__Theorem__ /42,43/. Under the $\widetilde{Z}_N$ reduction (8.1)-(8.2) we have

$$\omega^{(\ell)}(\delta_1 P, \delta_2 P)\Big|_{P = P_{\widetilde{Z}_N}} = 0$$

if $\ell \neq kN - 1$ $(k = 0, \pm 1, \pm 2, \ldots)$, and the set of closed symplectic forms

$$\omega_V^{(k)}(\delta_1 V, \delta_2 V) \stackrel{def}{=} \omega^{(kN-1)}(\delta_1 P, \delta_2 P)\Big|_{P = P_{\widetilde{Z}_N}} =$$

$$= \sum_{r,s=0}^{N-2} \int_{-\infty}^{+\infty} dx \, \delta_1 V_r(x) \, M_{rs}^{(k)} \, \delta_2 V_s \tag{8.7}$$

where $M_{rs}^{(k)}$ are definite integro-differential operators, determines the hierarchy of Hamiltonian structures for equations (8.5). In particular, $\omega_V^{(1)}$ is a symplectic form corresponding to the Gelfand-Dikij bracket (8.6).

__Proof.__ The first part of the theorem is verified directly. The explicit form of operators $M_{rs}^{(k)}$ is found by direct calculation. For example, at N = 2,

$$M_{00}^{(1)} \cdot = \frac{1}{2} \int_{-\infty}^{x} dy \cdot - \frac{1}{2} \int_{x}^{\infty} dy \cdot \quad \text{and} \quad \omega_V^{(1)}(\delta_1 V_0, \delta_2 V_0)$$

coincides with the well known symplectic form for the set of KdV equations /1,2/:

$$\omega_V^{(1)}(\delta_1 V_0, \delta_2 V_0) = \frac{1}{2} \int_{-\infty}^{+\infty} dx \, \delta_1 V_0(x) \Big(\int_{-\infty}^{x} dy \, \delta_2 V_0(y) - \int_{x}^{\infty} dy \, \delta_2 V_0(y) \Big) =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{x} dy \, \big(\delta_1 V_0(x) \, \delta_2 V_0(y) - \delta_2 V_0(x) \, \delta_1 V_0(y) \big) .$$

At N = 3 we have

$$M_{11}^{(1)} = M_{00}^{(1)} = 0 ,$$

$$M_{10}^{(1)} \cdot = M_{01}^{(1)} \cdot = -\frac{i}{6} \int_{-\infty}^{x} dy \cdot + \frac{i}{6} \int_{x}^{\infty} dy \cdot$$

The matrix $\widetilde{Z}_N$ - reduction is considered in a similar way. The connection between the matrix $\widetilde{Z}_N$ - reduction and the Gelfand-Dikij spectral problem is determined in an analogous manner, too /42/.

Similarly to the pure Z_N - and $\widetilde{Z}_N$ - reductions, the general Z_N reduction may be analysed.

IX. Some other linear reductions

Not all the linear constraints have the form (6.6). As an example, of the linear coupling not reduced to (6.6), let us consider the constraint*

$$P^T = -R P R^{-1} \tag{9.1}$$

where R is the constant matrix.

The reductions which are generated by the constraints (9.1) differ, in their form, from the reductions considered in section VI. However, for their analysis one can use the same, in its essence, procedure.

For the simplicity we assume that the matrix A is diagonal. The constraints (9.1) is consistent with the bundle (1.1) in the following cases:

$$\begin{aligned} &\alpha)\ R_\alpha A R_\alpha^{-1} = A , \quad R_\alpha \Psi^{-1}(-\lambda, x) R_\alpha^{-1} = \Psi^T(\lambda, x), \\ &\beta)\ R_\beta A R_\beta^{-1} = -A, \quad R_\beta \Psi^{-1}(\lambda, x) R_\beta^{-1} = \Psi^T(\lambda, x). \end{aligned} \tag{9.2}$$

Let us first consider the case (9.2α). We have $R_\alpha S^{-1}(-\lambda, t) =$

* The reductions of the type (9.1) have been considered by Mikhailov, too.

$= S^T(\lambda,t)\, R_\alpha$. The consistence of this equation with (3.5) gives $R_\alpha\, Y(-\lambda,t) R_\alpha^{-1} = -Y(\lambda,t)$. Hence $\Omega_n(-\lambda) = (-1)^{n+1}\, \Omega_n(\lambda)$ $(n = 1,\dots,N-1)$.

Thus equations (3.6) admit the reduction (9.1)-(9.2(α)) at the functions Ω_n of the form $\Omega_n = \lambda^{n+1}\Omega_n(\lambda^2)$.

In particular, at $R_\alpha = 1$, $P = -P^t$. Therefore, if $\Omega_n = \lambda^{n+1}\Omega_n(\lambda^2)$ and A is an arbitrary diagonal matrix, equations (3.6) admit the reduction to the algebra $SO(N,C)$. In this case independent variables is convenient to introduce as follows: $P = Q - Q^T$ where Q is the upper-triangular matrix with zeros on the main diagonal. In Ref. /9/ equations (3.6) have been rewritten in the form containing Q only and it has been shown that they are Hamiltonian with respect to the set of Poisson brackets

$$\{\mathcal{F},\mathcal{H}\}_n = \int_{-\infty}^{+\infty} dx\, \mathrm{tr}\left(\frac{\delta\mathcal{F}}{\delta Q^T}\left(L_Q^+\right)^n \mathcal{D}^+ \frac{\delta\mathcal{H}}{\delta Q}\right) \quad (n = 0,\pm1,\pm2,\dots) \tag{9.4}$$

where

$$\mathcal{D}^\pm := \frac{\partial}{\partial x} - i\left[Q-Q^T, \cdot\,\right]_{F(A)\Delta} \pm i\left(\left[Q-Q^T, \cdot\,\right]^T_{F(A)}\right)_\Delta$$

and

$$L_Q^+ \cdot = -\mathcal{D}^-\mathcal{D}_A^+ - 2\left[Q(x), \int_{-\infty}^{x} dy\,\left[Q(y)-Q^T(y), \mathcal{D}_A^+\cdot\,\right]_{0(A)}\right]$$

and Δ denotes the projection onto the upper triangular part of the matrix ($P_\Delta = Q$).

Note that under the reduction $P^T = -P$

$$\{\mathcal{F},\mathcal{H}\}_{2n}\Big|_{P=-P^T} = 0 \qquad (n = 0,\pm1,\pm2,\dots)$$

for any two functionals $\mathcal{F},\mathcal{H}$; $\omega^{2n}(\delta_1 P,\delta_2 P)\big|_{P=-P^T} = 0$ and the set of forms $\omega_Q^{(n)}(\delta_1 Q,\delta_2 Q) \overset{def}{=} \omega^{(2n+1)}(\delta_1 P,\delta_2 P)\big|_{P=-P^T}$ corresponds to the set of Poisson brackets (9.4).

At N = 2 the operator $\mathcal{D}^+ = \frac{\partial}{\partial x}$ and we obtain the equations

considered earlier in Ref. /29/.

Let us now consider the case (9.2β). Here $R_\beta S^{-1}(\lambda,t) = S^T(\lambda,t) R_\beta$, $R_\beta Y(\lambda,t) = -Y(\lambda,t) R_\beta$. It follows that $(-1)^n\Omega_n(\lambda) = -\Omega_n(\lambda)$ (n = 1,...,N-1). Hence all $\Omega_{2n} \equiv 0$ and the functions Ω_{2n+1} are arbitrary.

Thus equations (3.6) admit the reduction (9.1)-(9.2β) if $\Omega_{2n} \equiv 0$ $(n=1,\dots,[\frac{N}{2}])$ and functions Ω_{2n+1} are arbitrary.

Reduction (9.1)-(9.2β) contains, as particular cases, the reductions of the general equations (3.6) to classic algebras $SO(N,C)$ and $Sp(2N,C)$.

Indeed, algebras $SO(N,C)$ and $Sp(2N,C)$ may be separated from general linear algebra $gl(N,C)$ by the conditions

$$P^T = -\mathcal{Y} P \mathcal{Y}^{-1} \tag{9.5}$$

where $\mathcal{Y}$ are definite skew-symmetric matrices /18/. Assuming $R_\beta = \mathcal{Y}$ in (9.1) and (9.2), we obtain the reductions of equations (3.6) to classic algebras.

It is convenient to introduce the independent variables under these reductions as follows: $P = Q - \mathcal{Y}^{-1}Q^T\mathcal{Y}$ where Q is the matrix all the elements of which located below the side diagonal are zero. In Ref. /45/ equations (3.6) have been rewritten in the form which contains Q only. It has also been shown that these equations are Hamiltonian ones with respect to the set of Poisson brackets:

$$\{\mathcal{F},\mathcal{H}\}_n = \int_{-\infty}^{+\infty} dx\, \mathrm{tr}\left(\frac{\delta\mathcal{F}}{\delta Q}\left(L_Q^+\right)^n\left[A, \frac{\delta\mathcal{H}}{\delta Q}\right]\right) \tag{9.6}$$

where n is an arbitrary integer, and

$$L_Q^+\Phi = i\frac{\partial\Phi}{\partial x} + [P,\Phi]_\nabla + \left(\mathcal{Y}^{-1}[P,\Phi]^T\mathcal{Y}\right)_\nabla + i\left[Q(x), \int_{-\infty}^{x} dy\left(\left[P(y),\Phi(y)\right]_{0(A)} + \mathcal{Y}^{-1}\left[P(y),\Phi(y)\right]_{0(A)}\mathcal{Y}\right)\right] \tag{9.7}$$

where $P = Q - \mathcal{Y}\,\overset{-1\,T}{Q}\,\mathcal{Y}$ and ∇ is projection operation onto the subspace of matrices of the form $Q\,(P_\nabla = Q)$.

The reductions generated by the constraints $RP = P^+R$ and $RP = -P^*R$ where $+$, $*$ stand for the hermitian and complex conjugation may be considered in the same way as reductions (9.1). Note that for some concrete equations these reductions have been analysed in refs. /38,39/.

X. Conclusion

As we have seen the AKNS method is effective for the solution of two problems of the IST method, which have been formulated in the Introduction. This method enables one to describe, in a simple and convenient manner, the general form of integrable equations and to present the classes of equations which admit one or another reduction. Also this method allows to analyse the Hamiltonian structure of integrable equations both in the general position and under the reductions.

The results of the presented paper may be generalized in various directions. First, for the case of the bundle (1.1) with an arbitrary semi-simple matrix A (for $P_\infty = 0$ see /8/). One can consider the bundle (1.1) with Z_2 grading (the potential $P(x,t)$ contains both the commutating and anticommutating variables). The quadratic and arbitrary polynomial bundles (for $P_\infty = 0$ see /12/) may be analysed in a similar way.

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Appendix

We shall derive here the relations (2.13) and (4.2). Let us denote

$$\left(\overset{++}{\tilde{\Phi}}{}^{(in)}\right)_{kl} = (F^+)'_{kn}(F^+)^{-1}_{il}, \quad \left(\overset{-+}{\tilde{\Phi}}{}^{(in)}\right)_{kl} = (F^+)_{kn}(F^-)^{-1}_{il}.$$

From the equation (1.1) we have

$$\frac{\partial \tilde{\Phi}^{in}}{\partial x} = i\lambda\left[A, \tilde{\Phi}^{(in)}(x)\right] + i\,\left(P'(x)\,\tilde{\Phi}^{(in)}(x) - \tilde{\Phi}^{(in)}(x)\,P(x)\right). \quad (A.1)$$

Hence

$$\frac{\partial \tilde{\Phi}^{(in)}_{O(A)}}{\partial x} = i\left(P'(x)\,\tilde{\Phi}^{(in)}_{F(A)}(x) - \tilde{\Phi}^{(in)}_{F(A)}(x)\,P(x)\right)_{O(A)}. \quad (A.2)$$

From (A.2) we obtain

$$\tilde{\Phi}^{(in)}_{O(A)}(x) = \tilde{\Phi}^{(in)}_{O(A)}(+\infty) - i\int_x^\infty \left(P'\,\tilde{\Phi}^{(in)}_{F(A)} - \tilde{\Phi}^{(in)}_{F(A)}\,P\right)_{O(A)} \quad (A.3)$$

or

$$\tilde{\Phi}^{(in)}_{O(A)}(x) = \tilde{\Phi}^{(in)}_{O(A)}(-\infty) + i\int_{-\infty}^x \left(P'\,\tilde{\Phi}^{(in)}_{F(A)} - \tilde{\Phi}^{(in)}_{F(A)}\,P\right)_{O(A)} \quad (A.4)$$

where $\displaystyle\int_{-\infty}^x f \overset{def}{=} \int_{-\infty}^x dy\, f(y), \quad \int_x^\infty f \overset{def}{=} \int_x^\infty dy\, f(y).$

With the use of the asymptotic properties of F^+ and F^- and equality (2.1) we find that at $z \to \infty$

$$\left(\overset{++}{\tilde{\Phi}}{}^{(in)}(z)\right)_{kl} = D_{kn}(D^{-1})_{il}\, e^{i(\mu_n - \mu_i)z}, \quad (A.5)$$

$$\left(\overset{-+}{\tilde{\Phi}}{}^{(in)}(-z)\right)_{kl} = \sum_{m=1}^{N} S_{mn}\, D_{km}(D^{-1})_{il}\, e^{-i(\mu_m - \mu_i)z}. \quad (A.6)$$

Taking the projection of the equality (A.1) onto the subspace $\mathcal{G}_{F(A)}$ and taking into account (A.3), we obtain

$$\Lambda \overset{++}{\tilde{\Phi}}{}^{(in)}_{F(A)} = \lambda\left[A, \overset{++}{\tilde{\Phi}}{}^{(in)}_{F(A)}(x)\right] + P'(x)\,\overset{++}{\tilde{\Phi}}{}^{(in)}_{O(A)}(+\infty) - \overset{++}{\tilde{\Phi}}{}^{(in)}_{O(A)}(+\infty)\,P(x) \qquad (A.7)$$

$$(i, n = 1, \dots, N)$$

where

$$\Lambda\Phi = -i\frac{\partial\Phi}{\partial x} - \left(P'(x)\Phi(x) - \Phi(x)P(x)\right)_{F(A)} + P'(x)\int_x^\infty \left(P'\Phi - \Phi P\right)_{O(A)} - i\int_x^\infty \left(P'\Phi - \Phi P\right)_{O(A)}\cdot P(x) \qquad (A.8)$$

Let us consider the quantity $\overset{++}{\tilde{\Phi}}{}^{(in)}(\lambda, z)$. Because for large Z it is of the form (A.5), there always exist such values of indices i and n, which are not equal to each other, thet at least in the case of complex λ, $\mathrm{Im}(\mu_n - \mu_i) > 0$. As a result, for such i and n $\overset{++}{\tilde{\Phi}}{}^{(in)}(+\infty) = 0$. Let us denote the subspace of these quantities $\tilde{\Phi}^{(in)}$ through Φ^{Δ}. Hence

$$\Lambda \Phi^{\Delta}_{F(A)} = \lambda\left[A, \Phi^{\Delta}_{F(A)}(x)\right] \qquad (A.9)$$

i.e.

$$\Lambda_A \Phi^{\Delta}_{F(A)} = \lambda\, \Phi^{\Delta}_{F(A)}(\lambda, x). \qquad (A.10)$$

The explicit form of the operator Λ^+ is found by direct calculation. Note that $\Lambda^+_A \Phi \overset{def}{=} (\Lambda_A)^+\Phi = -\Lambda^+\Phi_A$.

Let us now derive the relation (4.2). Writting out equations of the type (A.1), (A.2), (A.3) for the quantity $(\tilde{\Phi}^{(in)})_{ke}$ it is not difficult to show that

$$L^+ \overset{-+}{\Phi}{}^{(in)}_{F(A)} = -\lambda\left[A, \overset{-+}{\Phi}{}^{(in)}_{F(A)}(x)\right] + \left[\overset{-+}{\Phi}{}^{(in)}_{O(A)}(-\infty), P(x,t)\right] \qquad (A.11)$$

where $L^+ = \Lambda^+(P' = P)$. As a result

$$\left(L^+_A - \lambda\right)\left[A, \overset{-+}{\Phi}{}^{(ii)}_{F(A)}\right] = -\left[\overset{-+}{\Phi}{}^{(ii)}_{O(A)}(-\infty), P(x,t)\right]. \qquad (A.12)$$

Taking into account that $\lim_{z\to\infty} \exp i(\mu_m - \mu_i)z = \delta_{im}$, from (A.6) we have

$$\left(\overset{-+}{\Phi}{}^{(ii)}(-\infty)\right)_{ke} = S_{ii}\, D_{ki}\left(D^{-1}\right)_{ie}. \qquad (A.13)$$

Multiplying the left- and right-hand sides of equality (A.12) by $(\bar{A}^n)_{ii}\frac{1}{S_{ii}}$, summing over i from 1 to N and taking into account equality (A.13) and relation $D\bar{A}D^{-1} = \tilde{A}$, we obtain the following relation

$$\left(L^+_A - \lambda\right)\left[A, \Pi^{(n)}\right] = -\left[(\tilde{A}^n)_{O(A)}, P(x,t)\right] \qquad (A.14)$$

where

$$\Pi^{(n)}_{ke} \overset{def}{=} \sum_{i=1}^{N} (\bar{A}^n)_{ii}\, \frac{\overset{-+}{\Phi}{}^{(ii)}}{S_{ii}}. \qquad (A.15)$$

Because $\tilde{A}^n = \sum_{m=0}^{n} \lambda^m T_{nm}$, we obtain (4.2).

We would like to emphasize that in derivation of the relations (A.1) and (A.10) we have not use the fact of commutativity of the elements of the matrix P at all. This means that these relations hold for the operator-valued elements of the matrix $P(x,t)$ as well.

References

1. "Solitons", Topics in Current Physics, v. 17, Eds. R.Bullough, P.Caudrey, Springer-Verlag, 1980.
2. V.E.Zakharov, S.V.Manakov, S.P.Novikov, L.P.Pitaevski, Soliton Theory. Method of the Inverse Problem, Ed. S.P.Novikov, Moscow, Nauka, 1980 (in Russian).
3. M.J.Ablowits, D.J.Kaup, A.C.Newell, H.Segur, Stud. Appl. Math., 53, 249 (1974).
4. I.Miodek, J. Math. Phys., 19, 19 (1978).
5. A.C.Newell, Proc. Roy. Soc. (London), A365, 283 (1979).
6. P.P.Kulish, Notes of LOMI scientific seminars, 96, 105 (1980); preprint LOMI P-3-79 (1979) (in Russian).
7. B.G.Konopelchenko, Phys. Lett., 75A, 447 (1980); preprint Institute of Nuclear Physics N 79-82 (1979).
8. B.G.Konopelchenko, Phys. Lett., 79A, 39 (1980).
9. B.G.Konopelchenko, J. Phys. A: Math. and Gen., 14, 1237 (1981); preprint Institute of Nuclear Physics N 80-16 (1980).
10. V.S.Gerdjikov, M.I.Ivanov, P.P.Kulish, Teor. Mat. Phys., 44, 342 (1980) (in Russian).
11. B.G.Konopelchenko, Phys. Lett., 95B, 83 (1980).
12. B.G.Konopelchenko, J. Phys. A: Math. and Gen., 14, N 11 (1981).
13. V.S.Gerdjikov, P.P.Kulish, Notes of LOMI scientific seminars, 101, 46 (1981) (in Russian).
14. A.V.Mikhailov, Pisma v ZETF, 30, 443 (1979); 32, 187 (1980) (in Russian).
15. A.V.Mikhailov, Proc. of Soviet-American Meeting on Soliton Theory, September 1979, Kiev, North-Holland P.C., 1980.
16. F.Calogero, A.Degasperis, J. Math. Phys., 22, 23 (1981).
17. A.Degasperis, Lecture Notes in Physics, v. 120, 16 (1980).
18. N.Bourbaki, Groups et Algebres de Lie, Hermann, Paris (1978).

19. J.-P.Serre, Lie Algebras and Lie Groups, New-York, (1966).
20. V.E.Zakharov, A.V.Mikhailov, Z.E.T.F., 74, 1953 (1978) (in Russian).
21. E.A.Goddington, N.Levinson, Theory of Ordinary Differential Equations, New-York, McGraw-Hill B.C., Inc. (1955).
22. F.R.Gantmaher, Matrix Theory, Moscow, Nauka (1967) (in Russian).
23. F.Calogero, A.Degasperis, Nuovo Cimento, 32B, 201 (1976).
24. M.Jaulent, J.H.Leon, Lett. Nuovo Cimento, 23, 129 (1978).
25. F.Magri, J. Math. Phys., 19, 1156 (1978).
26. P.P.Kulish, A.G.Reiman, Notes of LOMI scientific seminars, 77, 134 (1978) (in Russian).
27. I.M.Gelfand, I.Ya.Dorfman, Funct. Analysis and its Applications, 13, N 4, 13 (1979); 14, N 3, 71 (1980) (in Russian).
28. A.G.Reiman, M.A.Semenov-Tyan-Shansky, Funct. Analisys and its Applications, 14, N 2, 77 (1980) (in Russian).
29. H.Flaschka, A.C.Newell, in "Dynamical Systems, Theory and Applications", Ed. J.Moser, Lecture Notes in Physics, v. 38, p. 355 (1975).
30. V.I.Arnold, Mathematical Methods of the Classical Mechanics, Moscow, Nauka, 1974 (in Russian).
31. V.E.Zakharov, A.V.Mikhailov, Commun. Math. Phys., 74, 21 (1980).
32. E.K.Sklyanin, preprint LOMI E-3-79 (1979).
33. E.K.Sklyanin, Notes of LOMI scientific seminars, 95, 55 (1980) (in Russian).
34. V.E.Zakharov, S.V.Manakov, Teor. Mat. Phys., 19, 332 (1974); S.V.Manakov, Teor. Mat. Phys., 28, 172 (1976) (in Russian).
35. A.G.Izergin, V.E.Korepin, preprint LOMI E-3-80 (1980).
36. B.G.Konopelchenko, Phys. Lett., 100B, 254 (1981).
37. B.G.Konopelchenko, Proc. XVIII Winter School of Theor. Phys. in Karpacz, Poland, February-March 1981.

38. V.E.Zakharov, The Inverse Scattering Method, in "Solitons" Ref. /1/, p. 243 (1980).

39. V.E.Zakharov, A.B.Shabat, Funct. Anal. and its Applications, 13, N 3, 13 (1979) (in Russian).

40. B.G.Konopelchenko, preprint Institute of Nuclear Physics N 80-75 (1980) (in Russian).

41. P.A.Dirac, Lectures on Quantum Mechanics, New-York, Yeshiva University, 1964.

42. B.G.Konopelchenko, preprint Institute of Nuclear Physics N 80-223 (1980) (in Russian).

43. B.G.Konopelchenko, Funct. Anal. and its Applications, (in print) (in Russian).

44. I.M.Gelfand, L.A.Dikij, Funct. Anal. and its Applications, 11, N 2, 11 (1977); 12, N 2, 8 (1978) (in Russian).

45. B.G.Konopelchenko, V.G.Mokhnachev, J. Phys. A: Math. and Gen., 14, 1849, (1981); preprint Institute of Nuclear Physics, N 80-143 (1980).

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