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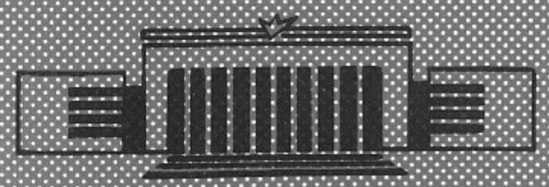

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ON THE QUANTUM CORRECTIONS TO THE STOCHASTIC MOTION OF THE NON-LINEAR OSCILLATOR

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A b s t r a c t

This paper considers the motion of a non-linear quantum oscillator influenced with an external force with a broad frequency spectrum. The time of the destruction of the classical phase trajectory by the quantum effects is exponentially small in the case in which the classical motion of the non-linear oscillator is stochastic. The approach is developed which takes into account all terms of the form $[\hbar\Delta^2(t)]^n$. These principal terms have an exponentially increased factor $\Delta^2(t)$ for every power of $\hbar$.

I. INTRODUCTION

This paper considers the motion of a non-linear quantum oscillator influenced with an external periodic force. The similar classical problem has been investigated by numerous authors and, as a result, a perfect clearness has been achieved (see, for example, Chirikov's survey articles[1]). In the case of the force with a broad frequency spectrum the behaviour of the oscillator depends significantly on a magnitude of the Chirikov's stochastisity parameter K[2]. The motion is a quite dynamic one when $K < 1$ but takes a stochastic character as $K > 1$. In particular, phases are random at $K > 1$ i.e. the phase correlations decrease exponentially with time.

A classical expression for the action $I_c(t)$ is of the form (the classical quantity is denoted by the subscription c)

$$I_c(t) = \int_0^t dt_1 \int_0^t dt_2 g(t_1)g(t_2)\cos[\varphi_c(t_1) - \varphi_c(t_2)] \qquad (1)$$

(For the sake of simplicity we choose a zero initial condition). The phases $\varphi_c(t)$ in equation (1) are a function of the action I_c : $\dot{\varphi}_c = \omega(I_c)$. For a carefully investigated case of the force which is a sequence of instantaneous impulses

$$g(t) = g_0 \sum_{s=-\infty}^{\infty} \delta(t - sT) \qquad (2)$$

the action $I_c(t)$ increases, on the average, linearly with time if $K = g_0^2 T \partial\omega/\partial T > 1$. This result is unchanged up to the time $t \sim T^2/K\tau_0$, as well as in the case when the impulses have a finite but small duration $\tau_0 \ll T$. The increasing of action is due to the range of values where $t_1 \approx t_2$ while region $t_1 \neq t_2$ determines action fluctuations.

Casati, Chirikov, Israilev and Ford [3] have raised a question of quantum-mechanic effects in stochastisity. Investigation of a plane rotator influenced with the force (2) has shown that the behaviour of the quantum system could be quite diffe-

rent from the classical one even at the high energy when a quasiclassical behaviour could be expected a'priori. In particular, a pure dynamic effect of quantum resonances was discovered and later this effect was investigated in detail by Israilev and Shepelansky [4] Also, the distinction of the time-dependence of energy on the classical case was achieved by means of numerical calculations. It's more or less clear that the first effect is connected with the specific time dependence of the force (2), while the second effect apparently has a fundamental character.

An attempt of analitical investigation of quantum effects in the stochastic motion of the non-linear oscillator was undertaken by Berman and Zaslavsky [5]. They have shown that the first correction proportional to $\hbar$ increases exponentially with time so that, as emphasized by the authors, the expansion over $\hbar$ becomes unapplicable very rapidly.

In this paper we will develop an approach which permits to analize an influence of quantum effects on the system motion up to large times. The approach is described in section II. In section III a generating function for average values of dynamical variables is introduced. In section IV the time-dependence of that variables is discussed on the example of the action $I(t)$ and a closed expression is derived for the action in the quantum case. It has a form suitable for comparison with the classical expression (1). Both quantum effects are considered in the case $K \ll 1$ when the classical motion is dynamic, and in the case $K \gg 1$ when it has a stochastic character. It is shown that quantum corrections in quasiclassical region are small at all times if $K \ll 1$ while at $K \gg 1$ they can exponentially increase with time. In section V we discuss shortly a a formal analogy of the developed approach with a classical theory of random processes and the reasons for which, in spite of this analogy, it is impossible to treat the approach as a theory with hidden variables.

II The statement of a problem and approach

Let us consider a non-linear oscillator described by the Hamiltonian

$$\hat{H} = \hat{H}_0 - \sqrt{\hbar}\, g(t)(\hat{a} + \hat{a}^+) \qquad (3)$$

where $g(t)$ is an external time-dependent force and

$$\hat{H}_0 = \hbar\, \omega_0 \hat{n} - \hbar^2 \gamma \hat{n}(\hat{n} - 1) \qquad (4)$$

is the Hamiltonian of the unperturbed non-linear oscillator characterized by non-linearity $\gamma (h\gamma \ll \omega_0)$ and $\hat{n} = \hat{a}^+\hat{a}$.

Operators $\hat{a}^+$ and $\hat{a}$ have in the basis of eigen-functions $|n\rangle$ of non-linear Hamiltonian $\hat{H}_0$ the same matrix elements as the usual creation and annihilation operators and satisfy the ordinary commutation relations for these operators. However, a connection of the operators $\hat{a}$, $\hat{a}^+$ with the operators of coordinate $\hat{x}$ and momentum $\hat{p}$ are more complicated than in the case of a linear oscillator (see details in ref. [6]).

The Hamiltonian (3) is expressed through the operators $\hat{a}$, $\hat{a}^+$ by analogy with the classical way when a Hamiltonian function is expressed through the angle-action variables. This Hamiltonian may be transformed into the classical Hamiltonian function

$$H_c = \omega_0 |\alpha_c|^2 - \gamma |\alpha_c|^4 - g(t)(\alpha_c + \alpha_c^*) =$$

$$= \omega_0 I_c - \gamma I_c^2 - 2g(t)\sqrt{I_c}\, \cos \vartheta_c \qquad (5)$$

by substitutions $\sqrt{\hbar}\,\hat{a} \to \alpha_c = \sqrt{I_c} e^{-i\vartheta_c}$; $\sqrt{\hbar}\,\hat{a}^+ \to \alpha_c^* = \sqrt{I_c} e^{i\vartheta_c}$

The connection of operators $\hat{a}$ and $\hat{a}^+$ of the non-linear oscillator with classical angle (ϑ_c) - action (I_c) variables becomes obvious if the coherent states of non-linear oscillator (see, for example, [6,7]) are used. Just as in the case of a li-

near oscillator they are eigen-functions of the annihilation operator :

$$\hat{a}|\alpha\rangle = \frac{\alpha}{\sqrt{\hbar}}|\alpha\rangle$$

and may be constructed from eigen-functions of Hamiltonian (4) as follows

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2\hbar}\right]\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\frac{\alpha}{\sqrt{\hbar}}\right]^n|n\rangle \qquad (6)$$

Unlike to the coherent states of the linear oscillator the functions (6) have not the Gaussian form in the coordinate or momentum representations, but an energy distribution has a Poisson character as before. It's important that the coherent states (6) are minimised the uncertainty relation[8] for operators $\hat{n}$ and

$$\hat{Cos}\,\theta = \frac{1}{2}\left[\frac{1}{\sqrt{\hat{n}+1}}\hat{a} + \hat{a}^+\frac{1}{\sqrt{\hat{n}+1}}\right]$$

if the action $I = h\langle\alpha|\hat{n}|\alpha\rangle$ is sufficiently large $I \gg \hbar$. Therefore, they describe, in the classical limit, oscillator states with definite values of the action I_c and angle ϑ_c, which corresponds to the point on the phase plane of variables I_c, ϑ_c or, that is the same, of variables α_c, α_c^*.

It's well known that description of quantum mechanical systems with the Vigner's function have common features with description of classical statistical systems with a distribution function. A density matrix in the coherent state basis $\langle\alpha|\hat{\rho}(\hat{a}^+,\hat{a};t)|\alpha\rangle \equiv \rho(\alpha^*,\alpha;t)$ is some modification of Vigner's function. The properties of the function $\rho(\alpha^*,\alpha;t)$ are in many respects the same as those for a classical distribution function: it is real, positive and satisfies the normalization condition:

$$\frac{1}{\pi\hbar}\int d^2\alpha\rho(\alpha^*,\alpha;t) = 1 \quad ; \quad d^2\alpha = dRe\alpha\cdot dIm\alpha$$

It is easy to show that $\rho(\alpha^*,\alpha;t) \le 1$, too.

The average value of operator $\hat{F}(\hat{a}^+,\hat{a})$ of arbitrary dynamic variable is expressed by means of $\rho(\alpha^*,\alpha;t)$ in the following way

$$\langle F\rangle = Sp\hat{F}(\hat{a}^+,\hat{a})\hat{\rho}(\hat{a}^+,\hat{a};t) = \frac{1}{\pi\hbar}\int d^2\alpha F^{(A)}(\alpha^*,\alpha)\rho(\alpha^*,\alpha;t),$$

where $F^{(A)}(\alpha^*,\alpha)$ is obtained according to the rule: $\hat{F}(\hat{a}^+,\hat{a})$ should be transformed into antinormal form and then $(1/\sqrt{\hbar})\alpha^*$ and $(1/\sqrt{\hbar})\alpha$ will be substituted instead of operators $\hat{a}^+$ and $\hat{a}$ correspondingly. Hence, it is clear that the plane (α,α^*) is analogous to the classic phase plane (α_c,α_c^*).

The general method of phase plane in quantum mechanics based on algebra of operators $\hat{a}_0^{(\pm)} \sim (\hat{x} \pm i\hat{p})$ has been developed by Agarwal and Wolf[9]. Because the commutation relations of the operators $\hat{a}$, $\hat{a}^+$ We used are the same as was assumed by Agarwal and Wolf, their results are applicable also in the case under consideration. In particular, following to [9] we have derived the equation for $\rho(\alpha^*,\alpha;t)$.

$$\frac{\partial}{\partial t}\rho(\alpha^*,\alpha;t) = \hat{L}(\alpha^*,\alpha;t)\rho(\alpha^*,\alpha;t) \qquad (7)$$

where $\hat{L}$ is of the form

$$\hat{L}(\alpha^*,\alpha;t) = -i\Big[(\omega_0 - 2\gamma|\alpha|^2)\Big(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\Big) -$$

$$- g(t)\Big(\frac{\partial}{\partial\alpha^*} - \frac{\partial}{\partial\alpha}\Big) - \hbar\gamma\Big(\alpha^{*2}\frac{\partial^2}{\partial\alpha^{*2}} - \alpha^2\frac{\partial^2}{\partial\alpha^2}\Big)\Big]$$

This equation is a quantum analogy to the Licuville equation for distribution function $f_c(\alpha^*,\alpha;t)$ and turn into it, if the only term proportional to $\hbar$ in the operator $\hat{L}$ will be omitted. Some initial condition should be formulated for Eq. (7). It's possible to describe the motion along a trajectory in the classical mechanics with the distribution function of the δ -

- functional form. The quantum-mechanical analogy of such a condition will be the density matrix constructed on a pure state $|\Psi(t)\rangle$. On the basis of the coherent states it is of the form

$$\rho(\alpha^*,\alpha;t) = |\langle\alpha|\Psi(t)\rangle|^2 = \exp\left(-\frac{|\alpha|^2}{\hbar}\right)|\Psi(\alpha;t)|^2 \quad (8)$$

where $\Psi(\alpha;t)$ is the wave-function at the α - representation. Unlike to the classical distribution function it is impossible that the function (8) be proportional to δ - function at $t = 0$ because of a finite magnitude of a phase cell in quantum mechanics. The condition $|\Psi(0)\rangle = |\overset{\circ}{\alpha}\rangle$ will be a natural analogy to the δ - function initial condition, i.e.

$$\frac{1}{\pi\hbar}\rho(\alpha^*,\alpha;0) = \frac{1}{\pi\hbar}\rho_{\overset{\circ}{\alpha}}(\alpha^*,\alpha;0) = \frac{1}{\pi\hbar}\exp\left(-\frac{|\alpha-\overset{\circ}{\alpha}|^2}{\hbar}\right) \quad (9)$$

This condition takes the δ - function form in the classical limit $\hbar \to 0$.

The general case of an arbitrary initial condition for eq. (7) is reduced to (9) with the help of Glauber's diagonal representation [10]:

$$\rho(\alpha^*,\alpha;0) = \frac{1}{\pi}\int d^2\overset{\circ}{\alpha} P(\overset{\circ}{\alpha}^*,\overset{\circ}{\alpha})\ \rho_{\overset{\circ}{\alpha}}(\alpha^*,\alpha;0)$$

It's clear that the solution of eq. (7) with initial condition (9) plays the role of the Green function on the plane (α^*,α).

III. The generating function for the average value of dynamical variables

It's convenient to proceed to the equation of wave function $\Psi(\alpha;t)$ related to $\rho(\alpha^*,\alpha;t)$ by eq. (8).

$$i\frac{\partial}{\partial t}\Psi(\alpha;t) = \hat{H}(\alpha;t)\Psi(\alpha;t)$$

$$\hat{H}(\alpha;t) = -\omega\alpha\frac{\partial}{\partial\alpha} + g(t)\left(\frac{1}{\hbar}\alpha + \frac{\partial}{\partial\alpha}\right) + \hbar\gamma\left(\alpha\frac{\partial}{\partial\alpha}\right)^2 \quad (10)$$

where is $\omega = \omega_0 + \hbar\gamma$. The equation has the formal solution of the form

$$\Psi(\alpha;t) = T\exp\{-i\int_0^t d\tau\hat{H}(\alpha;\tau)\}\Psi(\alpha;0) \quad (11)$$

We could represent the exponent under the chronological production with the help of the formula (see, for example, [11])

$$\exp\left\{-i\hbar\gamma\int_0^t d\tau\left(\alpha\frac{\partial}{\partial\alpha}\right)^2\right\} = \quad (12)$$

$$= \int D\lambda(\tau)\exp\left\{\frac{i}{4\hbar\gamma}\int_0^t d\tau\lambda^2(\tau) - i\int_0^t d\tau\lambda(\tau)\alpha\frac{\partial}{\partial\alpha}\right\}$$

The functional integrals are taken over all real $\lambda(\tau)$ and are normalized by condition

$$\int D\lambda(\tau)\exp\left\{\frac{i}{4\hbar\gamma}\int_0^t d\tau\lambda^2(\tau)\right\} = 1$$

The transformation (12) reduces the exponent operator (11) to the linear one over the derivative $\frac{\partial}{\partial\alpha}$. As a result, it's possible to represent (11) as

$$\Psi(\alpha;t) = \int D\lambda(\tau)\exp\left\{\frac{i}{4\hbar\gamma}\int_0^t d\tau\lambda^2(\tau)\right\}\chi_\lambda(\alpha;t) \equiv$$

$$\equiv \langle\chi_\lambda(\alpha;t)\rangle_\lambda^{(+)} \quad (13)$$

We are used the notation

$$\chi_\lambda(\alpha;t) =$$

$$= \exp\left\{-\frac{1}{\hbar}\left[S_\lambda(t)+iG_\lambda(t)\right]\right\}\Psi\left\{\exp\left[i\int_0^t d\tau(\omega-\lambda(\tau))\right]\left[\alpha - iG_\lambda^*(t)\right];0\right\} \quad (14)$$

$$G_\lambda(t) = \int_0^t d\tau g(\tau)\, \exp\left[i \int_\tau^t d\tau' (\omega - \lambda(\tau')) \right]$$

$$S_\lambda(t) = \int_0^t d\tau g(\tau) G_\lambda(\tau)$$

The brackets with subscriptions $(+)$ and λ in eq. (13) emphasizes that functional integration has a similarity with averaging over the Gaussian random noise $\lambda(\tau)$ with a correlator

$$\langle \lambda(\tau)\lambda(\tau') \rangle_\lambda^{(+)} = 2i\hbar\gamma\delta(\tau - \tau') \qquad (15)$$

The initial condition (9) means that

$$\Psi_{\mathring{\alpha}}(\alpha;0) = \exp\left[-\frac{1}{2}\frac{|\mathring{\alpha}|^2}{\hbar} + \frac{\mathring{\alpha}^*}{\hbar}\alpha \right]$$

For the sake of simplicity, we will confine ourselves to the special case $\mathring{\alpha} = 0$, $\Psi(\alpha,0) = 1$ assuming that the pulse at initial moment $t = 0$ gives the sufficiently large momentum to the oscillator. The nonzero initial conditions do not change significantly the results.

Under the chosen initial condition it is obtained from formulae (13) and (14) that

$$\rho(\alpha^*,\alpha;t) = \exp\left[-\frac{|\alpha|^2}{\hbar} \right] \left\langle \exp\left[-\frac{1}{\hbar}\left[S_{\lambda_1}(t) + i\alpha G_{\lambda_1}(t) \right] \right] \right\rangle_{\lambda_1}^{(+)} \times \qquad (16)$$

$$\times \left\langle \exp\left[-\frac{1}{\hbar}\left[S^*_{\lambda_2}(t) - i\alpha^* G^*_{\lambda_2}(t) \right] \right] \right\rangle_{\lambda_2}^{(-)}$$

where the symbol $\langle\,\rangle_\lambda^{(-)}$ means the averaging over the Gaussian random values $\lambda(\tau)$ with the correlator

$$\langle \lambda(\tau)\lambda(\tau') \rangle_\lambda^{(-)} = -2i\hbar\gamma\delta(\tau - \tau') \qquad (15^*)$$

or, in other words, the functional integration with same weight function is in the formula (13), but a complex conjugate one.

Below we will be interested in the quantum-mechanical average of operators, which are the functions of operator $\hat{n}$. Let us introduce the generating function ($0 \le \xi < 1$)

$$Z(\xi;t) = \frac{1}{\pi\hbar}\int d^2\alpha\, \exp\left[-\frac{\xi}{1-\xi}\frac{|\alpha|^2}{\hbar} \right] \rho(\alpha^*,\alpha;t) \qquad (17)$$

In particular, $Z(0;t)$ coincides with the normalization integral, the action is equal to

$$I(t) = \hbar\,\langle 0|\hat{n}(t)|0\rangle = \frac{1}{\pi\hbar}\int d^2\alpha(|\alpha|^2 - \hbar)\,\rho(\alpha^*,\alpha;t) = \qquad (18)$$

$$= -\hbar\left[\frac{\partial Z(\xi;t)}{\partial\xi}\bigg|_{\xi=0} + 1 \right]$$

and so on.

It's possible to carry out explicitly the integration over $d^2\alpha$ in (17) with density matrix (16) and after that only the functional integration over λ_1 and λ_2 remains. It is more suitable to use the lincar combinations

$$\lambda_1(\tau) = \mu_1(\tau) + \mu_2(\tau) \quad ; \quad \lambda_2(\tau) = \mu_1(\tau) - \mu_2(\tau) \qquad (19)$$

We include the constant Jacobian of the transformation in the normalization constant.

It's easy to see, that the correlators of μ_1, μ_2 are

$$\langle \mu_1(\tau)\mu_1(\tau') \rangle_{\mu_1,\mu_2} = \langle \mu_2(\tau)\mu_2(\tau') \rangle_{\mu_1,\mu_2} = 0$$
$$\langle \mu_1(\tau)\mu_2(\tau') \rangle_{\mu_1,\mu_2} = i\hbar\gamma\delta(\tau - \tau') \qquad (20)$$

$$Z(\xi;t) = (1-\xi)\left\langle \exp\left\{ \frac{1}{\hbar}\Phi_{12}(\xi;t) \right\} \right\rangle_{\mu_1,\mu_2} \qquad (21)$$

where

$$\Phi_{12}(\xi;t) = \int_0^t d\tau_1 \int_0^t d\tau_2 g(\tau_1)g(\tau_2)\mathrm{Cos}\left[\omega(\tau_1-\tau_2) - \int_{\tau_2}^{\tau_1} d\tau\mu_1(\tau)\right] \times$$

$$\times \left\{(1-\xi)\exp\left[-i\int_{\tau_1}^{t} d\tau\mu_2(\tau) - i\int_{\tau_2}^{t} d\tau\mu_2(\tau)\right] - \right. \qquad (22)$$

$$\left. - \theta(\tau_1-\tau_2)\exp\left[-i\int_{\tau_2}^{\tau_1} d\tau\mu_2(\tau)\right] - \theta(\tau_2-\tau_1)\exp\left[+i\int_{\tau_2}^{\tau_1} d\tau\mu_2(\tau)\right]\right\}$$

and $\theta(\tau)$ is the step function: $\theta(\tau) = 0$ at $\tau > 0$, $\theta(\tau) = 0$ at $\tau < 0$.

IV. The time-dependence of the action

In this section we show the time-dependence of the quantum-mechanical mean values on the example of the oscillator action dependence. In accordance with (18),(21) the action I(t) is the following

$$I(t) = \int_0^t dt_1 \int_0^t dt_2 g(t_1)g(t_2)e^{-i\omega(t_1-t_2)}\, Q(t;t_1,t_2)$$

where Q is

$$Q(t;t_1,t_2) = \langle \exp\left\{-i\int_0^t d\tau\left[\theta(\tau-t_1)-\theta(\tau-t_2)\right]\mu_1(\tau) - \right. \qquad (28)$$

$$\left. -i\int_0^t d\tau\left[\theta(\tau-t_1)+\theta(\tau-t_2)\right]\mu_2(\tau) + \frac{1}{\hbar}\Phi_{12}(t)\right\}\rangle_{\mu_1,\mu_2}$$

and $\Phi_{12}(0;t) \equiv \Phi_{12}(t)$.
We could not, in general, to derive an explicit expression for Q . There exists, however, the case when this is possible. If the force has the form of instantaneous impulses (2) and the condition $\hbar\gamma T = m\pi$ (m-integer) is fulfiled, then there is no difficulty in seeing after the change of variables in the functional integral (23)

$$\mu_1(\tau) \to \mu_1(\tau) + \hbar\gamma[\theta(\tau-t_1) + \theta(\tau-t_2)]$$

$$\mu_2(\tau) \to \mu_2(\tau) + \hbar\gamma[\theta(\tau-t_1) - \theta(\tau-t_2)]$$

that

$$Q(NT;s_1T,s_2T) = \exp[i\hbar\gamma(s_1-s_2)T]$$

($N \geq s_1, s_2$-integer). Really, the quantities $\mu(\tau)$ enter in the functional $\Phi_{12}(t)$ only through the integrals, which are arguments of the periodical functions. Therefore, it does not vary under the change of variables if the mentioned condition on T is fulfilled, so Q turns out to be proportional to the normalization integral. As a result, we have

$$I_N \equiv I(NT) = g_0^2 \frac{\mathrm{Sin}^2\left[N\frac{\pi}{2}\, m\omega_0/\hbar\gamma\right]}{\mathrm{Sin}^2\frac{\pi}{2}\, m\omega_0/\hbar\gamma}$$

This pure quantum-mechanical behaviour has the same origin as the quantum resonance discovered in ref.[3]. In particular, I_N increases quadratically with time if $\omega_0 = \frac{q}{m} 2\hbar\gamma$ (q - - integer).

Returning to the general case we represent Q in the form

$$Q(t;t_1,t_2) = \exp\{-R(t;t_1,t_2) + iJ(t;t_1,t_2)\}$$

It's easy to verify, making the complex conjugation and changing the variable $\mu_2(\tau) \to -\mu_2(\tau)$ that

$$R(t;t_2,t_1) = R(t;t_1,t_2) \quad ; \quad J(t;t_2,t_1) = -J(t;t_1,t_2)$$

Therefore,

$$I(t) = \int_0^t dt_1 \int_0^t dt_2 g(t_1)g(t_2)\exp[-R(t;t_1,t_2)] \times \qquad (24)$$

$$\times \mathrm{Cos}\left[\omega(t_1-t_2) - J(t;t_1,t_2)\right]$$

This expression must be transformed to expression (1) in the classical limit $\hbar \to 0$ and, moreover, $\varphi_c(t) = \omega_0 t - 2\gamma \int_0^t d\tau I_c(\tau)$ in the case of the Hamiltonian (5). Consequently, $J(t;t_1,t_2)$ has the classical limit equal to $2\gamma \int_{t_1}^{t_2} d\tau I_c(\tau)$ and R vanishes in this limit. Thus, the exponent in (24) depends on quantum effects completely.

As has been mantioned in the introduction, the region of small $\delta t_{12} = t_1 - t_2$ was significant in the classical case. Therefore we will concentrated on this region in the quantum case also. By virtue of anti-symmetry it is obvious that $J(t;t_1,t_2) = 0$. It's easy also to show that $R(t;t_1,t_1) = 0$. Really, the function $Q(t;t_1,t_2)$ in the case $t_1 = t_2$ is reduced by the linear shifting of the variable $\mu_1(\tau)$ to the normalised integral for a density matrix of the oscillator described with a Hamiltonian, which differs from the Hamiltonian (3) by the additional term $-\hbar^2\gamma\varepsilon(t_1 - t)\hat{n}$ where t_1 is some parameter and the step-function $\varepsilon(\tau)$ is determined as $\varepsilon(\tau)=1$, $\tau > 0$ and $\varepsilon(\tau) = -1$, $\tau < 0$. There exists also the direct proof of the statement.

When δt_{12} is sufficiently small, it is possible to expand J and R in a power series δt_{12}. We will show that

$$R(t;t_1,t_2) \approx \hbar\, \Delta^2(t_1)(\delta t_{12})^2 \; ; \; \Delta^2(t_1) = 2\gamma^2 I_c(t_1)|\Gamma(t_1)|^2 \quad (25)$$

Therefore, the contribution of the region $t_1 \neq t_2$ to the integral (24) are damped additionally in virtue of positiveness of the R. The magnitude of the $|\Gamma(t_1)|$ depends significantly on the Chirikov's stochastic parameter K. $|\Gamma(t_1)| \approx 1$ in the case $K \ll 1$ when the classical motion has a dynamic character, so that the quantum corrections remain small at all time. The quite different situation occurs if the $K \gg 1$ and the classical stochasticity takes place: the function $|\Gamma(t_1)|$ increases exponentially with time in some region of t_1.

Let us turn now to the proof of the statements. Suppose, that the amplitude of the force sufficiently large $g_0^2 \gg \hbar$ so that it is possible to evaluate the functional integral (23)

by the saddle-point method. It's useful, first of all, to evaluate the normalization integral $Z(0,t)$ by this method. The saddle-point conditions are of the form of the following system of integral equations:

$$\mu_1(\tau) =$$

$$= 2\gamma \int_0^\tau d\tau_1 \int_0^t d\tau_2 g(\tau_1)g(\tau_2)\cos\left[\omega(\tau_1 - \tau_2) - \int_{\tau_2}^{\tau_1} d\tau' \mu_1(\tau')\right] \times$$

$$\times\left\{\exp\left[-i\int_{\tau_1}^t d\tau' \mu_2(\tau') - i\int_{\tau_2}^t d\tau' \mu_2(\tau)\right] - \theta(\tau_2 - \tau_1)\exp\left[i\int_{\tau_2}^{\tau_1} d\tau' \mu_2(\tau')\right]\right\}$$

$$\mu_2(\tau) = \quad (26)$$

$$= -2i\gamma \int_0^\tau d\tau_1 \int_0^t d\tau_2 g(\tau_1)g(\tau_2)\sin\left[\omega(\tau_1 - \tau_2) - \int_{\tau_2}^{\tau_1} d\tau' \mu_1(\tau')\right] \times$$

$$\times\left\{\exp\left[-i\int_{\tau_1}^t d\tau' \mu_2(\tau') - i\int_{\tau_2}^t d\tau' \mu_2(\tau')\right] -\right.$$

$$\left. - \theta(\tau_1 - \tau_2)\exp\left[-i\int_{\tau_2}^{\tau_1} d\tau' \mu_2(\tau')\right] - \theta(\tau_2 - \tau_1)\exp\left[i\int_{\tau_2}^{\tau_1} d\tau' \mu_2(\tau')\right]\right\}$$

It has the solution $\mu_2(\tau) = 0$ and

$$\mu_1(\tau) = 2\gamma \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 g(\tau_1)g(\tau_2)\cos\left[\omega(\tau_1 - \tau_2) - \int_{\tau_2}^{\tau_1} d\tau' \mu_1(\tau')\right] \quad (27)$$

Comparison of eq. (27) with eq. (1) shows that $\frac{1}{2\gamma}\mu_1(t)$ coincides with the action $I_c(t)$ of the classical oscillator described by the Hamiltonian (5).

The function $\Phi_{12}(t)$ vanishes on the saddle-point trajectory (27), so that the magnitude of normalization integral is determined by a pre-exponential factor. It is easy to show that it is equal to unity. Thus, as it should be expected the saddle-point method gives the exact result for normalization integral.

Let us consider now the functional integral (23) which defines the function $Q(t;t_1,t_2)$. It differs from the normalization integral $Z(0,t)$ by an additional term in the exponent index without the factor $1/\hbar$. Therefore, it seems possible to extract this multiplier from the integral at the saddle-point trajectory (27). It's obvious that f.(24) coincides with the classical equation (1) in this approximation. If we take into account the multiplier mentioned above the saddle-point equation will differ from eq. (26) by additional terms
$\hbar\gamma[\theta(\tau-t_1)+\theta(\tau-t_2)]$ in the first equation and
$\hbar\gamma[\theta(\tau-t_1)-\theta(\tau-t_2)]$ in the second one respectively.
These terms shift the saddle-point trajectory proportionally to $\hbar$. However, the proportion coefficient could be strongly dependent on time. In this case the quantum effects could increase with time rapidly.

With the said above taken into account, we will decompose the solution of the new saddle-point equations in a power series of δt_{12}. In virtue of the symmetry the expansion of R starts with the second power of δt_{12}. If we carry out the expansion including the second power terms, we obtain the following:

$$J(t;t_1,t_2) \approx 2\gamma \int_{t_2}^{t_1} d\tau \, |a(\tau)|^2 = 2\gamma \int_{t_2}^{t_1} d\tau I_c(\tau)$$

$$R(t;t_1,t_2) \approx \hbar \cdot 2\gamma^2 \left| a(t_1) + \int_0^{t_1} d\tau a(\tau)\xi(\tau;t_1) \right|^2 (\delta t_{12})^2 \qquad (28)$$

Here the functions $a(\tau)$ and $\xi(\tau;t_1)$ are introduced:

$$a(\tau) = \int_0^{\tau} d\tau' g(\tau') \exp\left[i\omega\tau' - i \int_0^{\tau'} d\tau'' I_c(\tau'') \right] \qquad (29)$$

and $\xi(\tau;t_1)$ is the solution of the equation

$$\xi(\tau;t_1) = -4\gamma s(\tau;t_1) - 4\gamma \int_{\tau}^{t_1} d\tau' s(\tau,\tau')\xi(\tau';t_1) \qquad (30)$$

where

$$s(\tau,\tau') = \frac{1}{2i}\left[a(\tau)a^*(\tau') - a^*(\tau)a(\tau') \right]$$

in the time region $\tau < t_1$.

There is no difficulty in slowing that f.(28) could be rewritten by means of eq. (30) in the form (25) (see appendix). As shown in the appendix, the function $\Delta^2(t_1)$ under condition $K = \gamma g_0^2 T \gg 1$ and at large time $t_1 \gg T$ is of the form

$$\Delta^2(t_1) \approx 2\sqrt{2\pi}\, K \frac{\gamma}{T} \left(\frac{t_1}{T} \right)^2 \exp\left\{ 2\frac{t_1}{T} \ln\left[(4K)^2 \frac{t_1}{T} \right] \right\} \qquad (31)$$

The real parameter of the performed expansion is

$$\varkappa \equiv \int_0^{t_1} d\tau \mu_2(\tau) \approx \hbar\gamma\delta t_{12} \int_0^{t_1} d\tau \, \xi(\tau;t_1)$$

It's easy to verify that $\varkappa^2/R \sim \hbar\gamma T/K = \hbar/g_0^2 \ll 1$. Therefore, the expressions obtained are valid for the fairly large $R > 1$ inspite of $|\varkappa| \ll 1$.

It's possible to establish that the eigen-values of the secondary-derivatives matrix of functional (22) increase exponentially with time. Their product determines the pre-exponential factor in the saddle-point method. But the number of such terms is proportional to time $\sim t_1$ so that pre-exponent factor add to R the terms of order t_1^2. This changes the result insignificantly.

Consequently, expression (24) with R determined by (25), (31) takes into account all the terms of the form $[\hbar\Delta^2(t)]^n$ in the expansion of the action I over $\hbar$.

They are principal in the case of exponentially increased $\Delta(t)$. This result is considerably better than the usual quasi-classical approximation. It should be pointed out that it is impossible to achieve such a result for a wave-function directly.

At least, let us discuss the corrections which are generated by the following terms of the functional expansion near the saddle point. Remind that they do not contribute to the normalization integral. In the case, when the action I is evaluated, the saddle-point trajectory shifts by a magnitude of the order $\hbar\Delta(t_1)$ so that these corrections do not vanish. But they are proportional to the additional power of $\hbar$ and hence are negligibly small.

Let us clarify the influence of the quantum corrections on the time dependence of the action I(t). Since $R(t_1=t_2)=0$ it is clear that the contribution of the region $t_1=t_2$ is not changed by the quantum corrections in the case of the δ - - functional impulses. In this case the quantum corrections influence the terms, which describe the action fluctuations are stipulated by the complicated character of the classical phase trajectory. The influence is significant in the region $R \gtrsim 1$ which is achieved at $|\delta t_{12}| > T$ at the times

$$t^* \sim T \ln \left[\frac{1}{\hbar\gamma T \cdot K} \right] \qquad (32)$$

The estimate is true when the expansion over $\varkappa$ is valid, i.e.

$$\hbar\gamma T \, |\Gamma(t^*)| \ll 1 \qquad (33)$$

(this is possible only if $\hbar\gamma T \ll 1$).

However, in the case, when the impulses have the finite duration τ_0 , the contribution of the region $t_1 \approx t_2 \approx sT$ is unchanged until $\tau_0 \sqrt{\hbar}\,\Delta(sT) \ll 1$ but decreased by the factor $[\tau_0 \sqrt{\hbar}\,\Delta(sT)]^{-1}$ as long as $\tau_0 \sqrt{\hbar}\,\Delta(sT) \gtrsim 1$. Hence, in this case the quantum corrections slow down the exitation of the oscillator beginning with the time

$$t^{**} \sim T \ln \left[\left(\frac{T}{\tau_0} \right)^2 / \hbar\gamma T \cdot K \right] \qquad (34)$$

We could not say for how long time the tendency will remain

unchanged. The condition $|\varkappa| \ll 1$ is violated rather fastly and then it is necessary to solve the nonlinear saddle-point equations. It is hardly possible to find the solution analytically. Nevertheless, we want to emphasize that the behaviour of I(t) dependes significantly on the quantum corrections and is not determined by the classical phase trajectory, at least, at time $t > t^*$.

V. The possible interpretation of the method

As was mentioned above, the functional integration over λ_1, λ_2 are similar to the overaging over the random Gaussian values. However, the similarity was purely formal because of the complex weight function in the functional integrals. It seems attractive to choose the variables of integration in such a way that the integration is taken a sence of averaging of some classical quantity over the random variables.

First of all, we would like to notice the following. The solution of the equation (7) for the density matrix could be constructed as a functional integral of the form

$$\rho(\alpha^*,\alpha;t) =$$

$$= \int D\mu_1(\tau) \int D\mu_2(\tau) \exp\left\{ \frac{i}{\hbar\gamma} \int_0^t d\tau \mu_1(\tau)\mu_2(\tau) \right\} f_{12}(\alpha^*, \alpha; t) \qquad (35)$$

where the function $f_{12}(\alpha^*, \alpha; t)$ satisfies the following first - order equation:

$$\frac{\partial}{\partial t} f_{12} + i \frac{\partial}{\partial \alpha^*} \left\{ \left[\left(\omega_0 - \hbar\gamma - \mu_1(t) - 2\gamma|\alpha|^2 \right)\alpha^* - g(t) + \mu_2(t)\alpha^* \right] f_{12} \right\} -$$

$$- i \frac{\partial}{\partial \alpha} \left\{ \left[\left(\omega_0 - \hbar\gamma - \mu_1(t) - 2\gamma|\alpha|^2 \right)\alpha - g(t) - \mu_2(t)\alpha \right] f_{12} \right\} = 0 \qquad (36)$$

It's easy to verify the statement, if we make use of the transformation (12) and introduce the variables $\mu(t)$ with the shifted $\mu_1(\tau) \to \mu_1(\tau) + 2\hbar\gamma$

Equation (36) coincides formally with the Liouville equation for the distribution function of the nonlinear oscillator the classical motion of which is described by equations:

$$i\frac{d\alpha}{dt} = \left[\omega_0 - \hbar\gamma - \mu_1(t) - 2\gamma|\alpha|^2\right]\alpha - g(t) - \mu_2(t)\alpha$$

$$i\frac{d\alpha^*}{dt} = -\left[\omega_0 - \hbar\gamma - \mu_1(t) - 2\gamma|\alpha|^2\right]\alpha^* + g(t) - \mu_2(t)\alpha^* \qquad (37)$$

If the functional integration (35) could be considered as the averaging over the random variables $\mu(\tau)$ then eq. (7) would be the Focker-Plank equation for the nonlinear oscillator influenced with the random force.

Unfortunately, the similarity mentioned above has the formal character only. Really, the equations (37) could be considered as the equations of motion with the velocity dependent force in the case of the imaginary $\mu_2(\tau)$. However, we could not perform the needed turn of the integration contour in the complex plane μ_2 because the transformed functional integral becomes devergent. Instead of this, we could introduce the real variables r and χ given as follows:

$$\mu_1(\tau) = r(\tau)e^{i\chi(\tau)} \quad , \quad \mu_2(\tau) = ir(\tau)e^{-i\chi(\tau)} \quad ;$$

$$(0 \leqslant r < \infty \; ; \; 0 \leqslant \chi < 2\pi)$$

The functional integral will be transformed into the convergent integral with real weight function $\exp\left\{-\frac{1}{\hbar\gamma}\int_0^t d\tau r^2(\tau)\right\}$ but the equations (37) expressed in variables r, χ lost the sense of equations of motion.

These remarks illustrate the general assertion that it is impossible to treat the quantum mechanics as a theory with hidden variables.

VI. Conclusion

This paper describes the approach permitting to analyse the behaviour of the nonlinear oscillator dynamical variables at large time. The analysis is exemplified for the average $I(t) = \hbar\langle 0|\hat{n}(t)|0\rangle$.

This value is the quantum-mechanical analogy of the classical action. It is shown on this example that the magnitude of quantum corrections dependends significantly on the character of motion in the classical limit. In the case of dynamical classical motion of the oscillator the quantum corrections remain small, but they increase exponentially with time if the classical motion becomes stochastic. This means that at the time $t > t^*$ (see f.(34)) the time-dependence of action differs significantly from the classical dependence if $K \gg 1$. The same is valid for the other quantum-mechanical averages.

However, the consideration of the time-dependence of the quantum-mechanical averages is insufficient, of course, to clear up the question about the dynamic or stochastic character of the quantum oscillator motion with $K \gg 1$. To answer this question it is necessary to such properties of the system as the quasi-energy spectrum and the character of the phase correlations. The authors intend to discuss these problems elsewhere.

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A p p e n d i x

We introduce the value $\Gamma(\tau;t)$ by the formula

$$a(t_1) + \int_{\tau}^{t} d\tau' a(\tau')\xi(\tau';t_1) = a(t_1)\Gamma(\tau;t_1)\exp\left\{i\left[\sigma_c(t_1) - \sigma_c(\tau)\right]\right\} \qquad (\Pi.1)$$

where $-\sigma_c(\tau)$ is the phase of $a(\tau)$. It is clear that $\Gamma(\tau=0,t_1) \equiv \Gamma(t_1)$. In virtue of ($\Pi.1$), eq. (28) is transformed into the eq. (25). It's easy to see that eq. (30) is equivalent to the next equation

$$\frac{d}{d\tau}\Gamma(\tau;t_1) = i\Gamma(\tau;t_1)\frac{d}{d\tau}\sigma_c(\tau) - 4\gamma I_c(\tau)\mathrm{Im}\,\Gamma(\tau;t_1) \qquad (\Pi.2)$$

with the additional condition $\Gamma(t_1;t_1) = 1$

We represent then the value $\Gamma(\tau;t)$ in the form $\Gamma(\tau;t_1) = |\Gamma(\tau;t_1)|\exp\{i\Lambda(\tau;t_1)\}$ and separate the real and maginary parts of eq. ($\Pi.2$). As a result, the connection

$$|\Gamma(t_1)| = \exp\left\{2\gamma\int_0^{t_1} d\tau I_c(\tau)\mathrm{Sin}\,2\Lambda(\tau;t_1)\right\} \qquad (\Pi.3)$$

and the differential equation

$$\frac{d}{d\tau}\Lambda = \frac{d}{d\tau}\sigma_c(\tau) \pm 4\gamma I_c(\tau)\mathrm{Sin}^2\Lambda(\tau) \qquad (\Pi.4)$$

for the function $\Lambda(\tau;t_1)$ will be obtained with the additional condition $\Lambda(t_1;t_1) = 0$.

The τ - dependence of the function $\Lambda(\tau;t_1)$ is varied unessentially when duration of impulse τ_0 is changed if $\tau_0 \ll T$. Hence, we will estimate the magnitude of $|\Gamma(t_1)|$ with the δ - functional form of the impulses ($\tau_0 = 0$). Both $\sigma_c(\tau)$ and $I_c(\tau)$ are changed at the moment of the impulse only. With the help of the classical equations of motion it is easy to show that the impulse acting at the moment sT leads to the following change of the value σ :

$$\mathrm{tg}(\sigma_s^+ - \sigma_s^-) \equiv \mathrm{tg}\,\nu_s = -\frac{g_0}{\sqrt{I_s^-}}\mathrm{Cos}\,\vartheta_s^-\left(1 - \frac{g_0}{I_s^-}\mathrm{Sin}\,\vartheta_s^-\right)^{-1}$$

The superscript $-$, $+$ denotes the action I_c and angle ϑ_c before and immediately after the impulse. For the simplicity of notation we omit here the subscript "c" for the classical variables.

The equation (32) shows that $\Lambda \equiv 0$ if $\sigma_c \equiv 0$ therefore the variation of Λ is stipulated by the phase step of the value $a(\tau)$. The step of Λ is equal to that for σ_c at the moment of the impulse and the change of Λ is described by the homogeneous equation between the impulses. The latter may be solved explicity. Representing the integral in the exponent (31) as the sum of the integrals over the time between the impulses, we obtain

$$|\Gamma(t_1 = NT + 0)| = \left|\prod_{S=0}^{N-1}\frac{\mathrm{Sin}\,\Lambda_{s+1}^-}{\mathrm{Sin}\,\Lambda_s^+}\right| = \left|\frac{\mathrm{Sin}\,\Lambda_N^-}{\mathrm{Sin}\,\Lambda_0^+}\prod_{S=1}^{N-1}\frac{\mathrm{Sin}\,\Lambda_s^-}{\mathrm{Sin}\,\Lambda_s^+}\right| \qquad (\Pi.5)$$

Since the additional condition for Λ is given at $\tau = t_1 = NT + 0$ the solution of the equation (32) should be constructed from the large τ to the small one. Under condition of stochasticity of classical motion $K \gg 1$ and for $s \gg 1$, $N = t_1/T \gg 1$ the value I_s^- of order $I_s^- \simeq g_0^2 s$ and the step σ_c is small $\nu_s \approx \mathrm{Cos}\,\vartheta_s^-/\sqrt{s}$, $|\nu_s| \ll 1$. At the same time, the variation of Λ between the impulses is large, so in virtue of (32)

$$\mathrm{ctg}\,\Lambda_s^+ = \mathrm{ctg}\,\Lambda_{s+1}^- + 4\gamma I_{s+1}^- T \qquad (\Pi.6)$$

and also $4\gamma T I_{s+1}^- \simeq 4K(s+1) \gg 1$. As a result, we are found that $\Lambda_s^- \approx q\pi + \mathrm{Cos}\,\vartheta_s^-/\sqrt{s}$ and that time $\Lambda_s^+ \approx q\pi + \frac{1}{4K(s+1)}$ when q - integer ($q > 0$ at $\gamma < 0$ and $q < 0$ at $\gamma > 0$). Consequently,

$$\frac{\mathrm{Sin}\,\Lambda_s^-}{\mathrm{Sin}\,\Lambda_s^+} \sim 4K\frac{s+1}{\sqrt{s}}\mathrm{Cos}\,\vartheta_s^-$$

If we take into account $\mathrm{Sin}\,\Lambda_N^- = -\mathrm{Sin}\,\nu_N \sim (g_0/\sqrt{I_N^-})\mathrm{Cos}\,\vartheta_N^-$ and set $\sin\Lambda_0^+ \sim 1/4K$ we obtain from (П.5)

$$|\Gamma(NT+0)| \approx \frac{g_0}{\sqrt{I_N^-}}\,(4K)^N\,N\sqrt{(N-1)!}\,\prod_{s=1}^{N}|\mathrm{Cos}\,\vartheta_s^-|$$

The formula (31) follows from this expression squred, by replacement $\mathrm{Cos}^2\vartheta_s^- \to \frac{1}{2}$ on account of the fact the phases are random, and the use of Stirling approximation for the factorial.

Otherwise, if $K \ll 1$ the action $I_s^- \sim g_0^2$ at all time s, the phase step $|\nu_s| \sim 1$ and the variation of Λ between the impulses is small

$$\Lambda_s^+ \approx \Lambda_{s+1}^- - 4\gamma T I_{s+1}^-\,\mathrm{Sin}^2\,\Lambda_{s+1}^-$$

Therefore,

$$|\Gamma(NT+0)| \approx 1 + 2\gamma T\sum_{s=0}^{N-1} I_{s+1}^-\,\mathrm{Sin}\,(2\sum_{q=s}^{N}\nu_q)$$

i.e. $|\Gamma|$ is given by the sum of oscillating terms, which do not increase with time N.

The estimate (П.7) is checked by numerous integration of eq. (30). The result of calculation of the value $|\Gamma(NT)|^2$ divided by the value given by (П.7) is represented in table I for various K,N. The set of random phases φ_c had been given and then the result is averaged the large number (10, 100, 1000) of such sets. The result slightly depends on the number of the sets.

As it is obvious from the table I the ratio obtained is of order unity, although the value $|\Gamma|^2$ which is determined by (П.7) varies by a magnitude of order 10^{15} (compare with table II, where $\log|\Gamma|^2$ is given).

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The table I

The ratio $|\Gamma/\Gamma_0|^2$ averaged over the 1000 sets of phases φ_s, s=1,...,5.

K \ N	I	2	3	4	5
2	1.076	1.321	1.514	1.979	2.646
3	1.024	1.066	1.635	1.635	2.738
4	1.031	1.086	1.224	1.759	2.293
5	1.005	0.936	1.117	1.117	2.143
6	0.970	0.943	1.300	1.648	2.205
7	0.999	1.034	1.236	1.615	2.492

The table II

The logarithm $\lg|\Gamma|^2$

K \ N	I	2	3	4	5
2	1.51	3.31	5.29	7.40	9.60
3	1.86	4.02	6.35	8.81	11.4
4	2.11	4.52	7.10	9.81	12.6
5	2.30	4.90	7.68	10.6	13.6
6	2.46	5.22	8.16	11.2	14.4
7	2.59	5.49	8.56	11.8	15.0

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