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ON THE STRUCTURE OF EQUATIONS INTEGRABLE BY THE ARBITRARY-ORDER LINEAR SPECTRAL PROBLEM

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Abstract

The general form of partial differential equations integrable by the arbitrary-order linear spectral problem is found. The groups of Backlund transformations corresponding to these equations are constructed. It is shown that partial differential equations of the class under study are Hamiltonian ones. Some reductions of general equations are considered. In Particular, the Hamiltonian structure of the generalisations of the sine-Gordon equation to the groups $GL(N)$, $SU(N)$ and $SO(N)$ at arbitrary N is proved.

1. INTRODUCTION

The inverse spectral transform (IST) method allows a comprehensive study of a great number of various partial differential equations (see, e.g. Refs./1-3/). The general scheme of this method was discussed in Refs./4,5/.

All the differential equations to which the IST method is applicable are united in the classes of equations integrable by the same linear spectral problem. The simple and convenient description of the class of equations which are integrable with the help of the second-order linear (in spectral parameter) spectral problem was presented in Ref./6/. This class of equations is characterized by the (n-1)th arbitrary functions (n is the number of independent variables) and by a certain integro-differential operator /6,7/. The analogous results were obtained for the class of equations which are associated with the matrix stationary Schrödinger equation /8/, the general linear spectral problem of arbitrary order /9-11/, with the second-order linear problem quadratic in its spectral parameter /12/ and with the general arbitrary-order linear spectral problem polynomial with respect to spectral parameter /13/. Within the framework of this approach the wide classes of Backlund transformations (BTs) which are playing a significant role in a study of nonlinear differential equations are also found /7,8,11,13/. For equations integrable by the second-order linear problem, the Hamiltonian structure of all the equations of thisclass is analysed /14/.

In the present paper we are going to study a class of partial differential equations connected to the general linear spectral problem of arbitrary order:

$$\frac{\partial \psi}{\partial x} = \left(i\lambda A + iP(x,t,\ldots)\right)\psi \qquad (1.1)$$

where λ is the spectral parameter, A the constant diagonal matrix ($A_{ik}=a_i\delta_{ik}$, $a_i \neq a_k$, $i,k=1,\dots,N$), the "potentials" $P(x,t,\dots)$ are the matices $N\times N$. It is not assumed here, in contrast to papers /9/ and /11/, that $P_{ii}=0$ $(i=1,\dots,N)$.

We find the general form of equations integrable by means of Eq.(1.1) and construct the Backlund transformations corresponding to these equations. As it will be seen, BTs and integrable equations are closely connected to the group of transformations conserving the form of spectral problem (1.1).

It is shown in this paper that the equations integrable with the help of Eq.(1.1) are Hamiltonian ones both in the general case and in the cases when the "potentials" satisfy the relations $P^+=P$ and $P_T=-P$. The case of the singular dispersion law is also studied. Among such equations there are the relativistic-invariant equations coinciding with the non-Abelian generalizations of the sine-Gordon equation /15,16/, which, as known, are gauge-equivalent to the equations of the principal chiral field /16,5/. The Hamiltonian structure of these equations for the groups $GL(N)$, $SU(N)$ and $SO(N)$ with arbitrary N is proved. For all equations considered in this paper, the uniqueness of the symplectic structure (similarly to the case $N=2$ /17,18/) takes place.

We mainly use the compact matrix notation proposed in Ref./9/. Let us remind them: for an arbitrary matrix Q the matrices Q_F and Q_D are determined as follows:

$(Q_F)_{ik}=Q_{ik}$ at $i\neq k$ $(i,k=1,\dots,N)$; $(Q_F)_{ii}=0$ $(i=1,\dots,N)$,

$(Q_D)_{ik}=0$ at $i\neq k$ $(i,k=1,\dots,N)$; $(Q_D)_{ii}=Q_{ii}$ $(i=1,\dots,N)$.

The matrix Q_R is given by the relation $[A,Q_R]=Q$, i.e.

$$(Q_R)_{ik}=\frac{1}{a_i-a_k}Q_{ik} \qquad (i\neq k,\ i,k=1,\dots,N).$$

The paper is organized as follows. The form of transformations of the transition matrix and potentials P conserving the spectral problem (1.1) is found in Section 2. The way by which the integrable equations and Backlund transformations are connected to these transformations is shown in section 3. The equations with singular dispersion law and, in particular the relativistic-invariant equations, are considered in section 4. The fifth section is devoted to the Hamiltonian structure of integrable equations with $P\in$ algebra $GL(N)$ and $P\in$ algebra $SU(N)$.

Equations with $P_T=-P$ are examined in section 6. Their Hamiltonian structure is proved in the last, seventh section. In particular, the Hamiltonian structure of the generalisation of the sine-Gordon equation to the group $SO(N)$ with arbitrary N is proved, and the explicit form of the Hamiltonian is found.

II. The group of transformations conserving the linear spectral problem

Let us examine an arbitrary transformation $P\to P'$, $\psi\to\psi'$, which conserves the mapping $P(x,t)\leftrightarrow\psi(x,t,\lambda)$ given by the set of linear differential equations (1.1).

It's easy to see that

$$\psi'-\psi K=-i\psi\int_x^\infty dy\,\psi^{-1}(P'-P)\psi' \tag{2.1}$$

where the constant matrix K is determined by the asymptotic properties of the matrices-solutions ψ .

It is assumed that $P(x,t,\dots)\to 0$ at $|x|\to\infty$.* Then

* The case $P(x)\underset{|x|\to\infty}{\longrightarrow}P_0$, where P_0 is the constant diagonal matrix, is reduced to this by the transformations $\psi\to exp(-iP_0x)\psi$ and $P\to exp(-iP_0x)(P-P_0)\,exp\,iP_0x$.

$\psi \xrightarrow[|x|\to\infty]{} E = \exp i\lambda A x$. Let us introduce, following to /19/, fundamental matrices-solutions F^+, F^- with asymptotics $F^{\pm} \xrightarrow[x\to\pm\infty]{} E$ and the transition matrix S : $F^+(x,t,\lambda) = F^-(x,t,\lambda)S(\lambda,t)$.

Setting $\psi = F^+$ and coming in Eq.(2.1) to the limit $x \to -\infty$, we get

$$S' - S = -i S \int_{-\infty}^{+\infty} dx\, F^{+-1}(P' - P)F'^{+} \tag{2.2}$$

Formula (2.2) which relates a change of the potentials P to a change of the transition matrix is a basis for our further discussion.

Let us suppose that the transition matrix at $P \to P'$, $\psi \to \psi'$ is transformed as

$$S \to S' = B^{-1} S C \tag{2.3}$$

where B and C are the diagonal matrices independent of the variable x . Rewriting Eq.(2.3) in the form $S' - S = (1-B)S' - S(1-C)$ and comparing it with Eq.(2.2), we have

$$\{S^{-1}(1-B)S'\}_F = -i\int_{-\infty}^{+\infty} dx \{F^{+-1}(P'-P)F'^{+}\}_F \tag{2.4}$$

$$\{S^{-1}(1-B)S'\}_{\mathcal{D}} - 1 + C = -i\int_{-\infty}^{+\infty} dx \{F^{+-1}(P'-P)F'^{+}\}_{\mathcal{D}}. \tag{2.5}$$

It follows from formulae (2.4) and (2.5) that transformations (2.3) are given by the matrix B which can be arbitrary. Matrix C is determined by equality (2.5).

Taking into account the relation

$$\{S^{-1}(1-B)S'\}_F = -\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}\{F^{+-1}(1-B)F'^{+}\}_F =$$

$$= i\int_{-\infty}^{+\infty} dx \{F^{+-1}\{P(1-B) - (1-B)P'\}F'^{+}\}_F$$

we get

$$\int_{-\infty}^{+\infty} dx \{F^{+-1}(BP' - PB)F'^{+}\}_F = 0 \tag{2.6}$$

Rewriting Eq.(2.6) by components and introducing the notation $\overset{++}{\Phi}{}^{(in)}_{ke} = (F^{+-1})_{ik}(F'^{+})_{en}$, we have $\quad (B_{ik}(\lambda) = B_i(\lambda)\delta_{ik})$

$$\int_{-\infty}^{+\infty} dx \sum_{k,e}\left(B_k(\lambda)P'_{ke}(x,\dots) - B_e(\lambda)P_{ke}(x,\dots)\right)\overset{++}{\Phi}{}^{(in)}_{ke}(x,\dots,\lambda) = 0 \tag{2.7}$$

$$(i \neq n,\ i,n = 1,\dots,N)$$

Formula (2.7) contains the product $B(\lambda)\,\Phi^{(in)}(\lambda)$ which is given in a local manner, at each point λ of the bundle (1.1). The spectral problem (1.1) makes it possible to transform this local product into the global product determined already on the whole bundle.

As shown in Appendix 1, in a space which covers all the non-diagonal quantities $\overset{++}{\Phi}{}^{(in)}_{ke}$ $(i\neq n,\ k\neq e;\ i,k,e,n = 1,\dots,N)$, the relation holds

$$\Lambda_R \overset{++}{\Phi}{}^{(in)}_{F} = \lambda \overset{++}{\Phi}{}^{(in)}_{F} \qquad \begin{matrix} i\neq n \\ i,n = 1,\dots,N \end{matrix} \tag{2.8}$$

where

$$\Lambda\Phi = i\frac{\partial\Phi}{\partial x} + \left(\Phi P'_T(x) - P_T(x)\Phi\right)_F -$$

$$- i\Delta^{-1}(x)\int_x^{\infty} dy\, \Delta(y)\left(\Phi(y)P'_T(y) - P_T(y)\Phi(y)\right)_{\mathcal{D}} P'_{TF}(x) \tag{2.9}$$

$$+ i P_{TF}(x)\Delta^{-1}(x)\int_x^{\infty} dy\, \Delta(y)\left(\Phi(y)P'_T(y) - P_T(y)\Phi(y)\right)_{\mathcal{D}}$$

Here and below P_T stands for the transposed matrix P . Thus, for entire functions $B_k(\lambda)$

$$\sum_{\substack{q,p \\ (q\neq p)}} \{B_k(\Lambda_R)\}_{keqp} \overset{++}{\Phi}{}^{(in)}_{qp} = B_k(\lambda)\overset{++}{\Phi}{}^{(in)}_{ke} \tag{2.10}$$

By virtue of this, equality (2.7) may be written as (one must extract the contribu-ion of diagonal quantities $\overset{\pm\pm}{\Phi}{}^{(n)}_{kk}$ and take into account Eq.(A1.3))

$$\int_{-\infty}^{+\infty} dx \sum_{\substack{q,p\\(q\neq p)}} \Big\{ \sum_{\substack{k,e\\(k\neq e)}} \Big\{ P'_{ke}(x) \{B_k(\Lambda_R)\}_{keqp} - P_{ke}(x) \{B_e(\Lambda_R)\}_{keqp} \Big\} \overset{\pm\pm}{\Phi}{}^{(in)}_{qp} -$$

$$- i \sum_k (P'_{kk} - P_{kk}) \Delta_k^{-1}(x) \int_x^{\infty} dy \, \Delta_k(y) \sum_{p\neq k} \sum_{m,q} \Big\{ P'_{kp}(y) \{B_k(\Lambda_R)\}_{kpmq} \overset{\pm\pm}{\Phi}{}^{(in)}_{mq} \tag{2.11}$$

$$- P_{pk}(y) \{ B_k(\Lambda_R) \}_{pkmq} \overset{\pm\pm}{\Phi}{}^{(in)}_{mq} \Big\}\Big\}\Big\} = 0$$

where
$$\Delta(x) = \exp i \int_{-\infty}^{x} dy \big(P_{\mathcal{D}}(y) - P'_{\mathcal{D}}(y) \big).$$

Integrating Eq.(2.11) by parts and changing the order of integration, i.e. making the transition from the operator Λ to the adjoint operator Λ^+, we get

$$\int_{-\infty}^{+\infty} dx \sum_{m,q} \sum_{k,e} \overset{\pm\pm}{\Phi}{}^{(in)}_{mq}(x) \Big(\{ B_k(\Lambda_R^+) \}_{mqke} P'_{ke} \Delta_k -$$
$$- \{ B_e(\Lambda_R^+) \}_{mqke} P_{ke} \Delta_e \Big) = 0 \tag{2.12}$$

where
$$\Lambda^+ \Phi = -i \frac{\partial \Phi}{\partial x} + \big(\Phi P'(x) - P(x) \Phi \big)_F$$
$$- i \Delta(x) \int_{-\infty}^{x} dy \, \Delta^{-1}(y) \big(\Phi(y) P'_F(y) - P_F(y) \Phi(y) \big)_{\mathcal{D}} P'(x) \tag{2.13}$$
$$+ i P(x) \Delta(x) \int_{-\infty}^{x} dy \, \Delta^{-1}(y) \big(\Phi(y) P'_F(y) - P_F(y) \Phi(y) \big)_{\mathcal{D}} .$$

Formula (2.12) is a relation between F^+, F'^+, P and P' in the transformations conserving the spectral problem (1.1). Equality (2.12) is fulfilled, if the expression in parentheses is equal to zero. Hence, the transformations $P \to P'$ conserving Eq.(1.1) are

of the form

$$\sum_{\substack{k,e\\(k\neq e)}} \Big(\{ B_k(\Lambda_R^+) \}_{mnke} P'_{ke} \Delta_k - \{ B_e(\Lambda_R^+) \}_{mnke} P_{ke} \Delta_e \Big) = 0 \tag{2.14}$$
$$m \neq n, \quad m,n = 1, \dots, N$$

Remind that $B_k(\lambda)$ are the arbitrary entire functions.

Transformation properties of the transition matrix are determined by formula (2.3). The transformation law of S' can be represented in a more explicit form. Let us write out Eq.(2.) as

$$S' - S = - i S I + S' \{ S^{-1} (1 - B) S' \}_F \tag{2.15}$$

where
$$I = \int_{-\infty}^{+\infty} dx \{ F^{-1+} (P' - P) F'^+ \}_{\mathcal{D}} .$$

With allowance for the relation (see Appendix 1)

$$(\Lambda_R - \lambda) \overset{\pm\pm}{\Phi}{}^{(nn)}_F = \{ P_{TF}(x) \Delta(+\infty) \Delta^{-1}(x) \overset{\pm\pm}{\delta}{}^{nn}$$
$$- \Delta(+\infty) \Delta^{-1}(x) \overset{\pm\pm}{\delta}{}^{nn} P'_{TF}(x) \}_R \tag{2.16}$$

we have

$$I_{nn} = \int_{-\infty}^{+\infty} dx \, tr \Big\{ (P'_T - P_T)(\Lambda_R - \lambda)^{-1} \big(P_{TFR}(x) \Delta(+\infty) \Delta^{-1}(x) \overset{\pm\pm}{\delta}{}^{nn}$$
$$- \Delta(+\infty) \Delta^{-1}(x) \overset{\pm\pm}{\delta}{}^{nn} P'_{TFR}(x) \big) \Big\} + \int_{-\infty}^{+\infty} dx \, tr \Big\{ (P'_T(x) - P_T(x)) \Big[\Delta(+\infty) \Delta^{-1}(x) \overset{\pm\pm}{\delta}{}^{nn}$$
$$- i \Delta^{-1}(x) \int_x^{\infty} dy \Big[(\Lambda_R - \lambda)^{-1} \big(P_{TFR}(y) \Delta(+\infty) \Delta^{-1}(y) \overset{\pm\pm}{\delta}{}^{nn} - \Delta(+\infty) \Delta^{-1}(y) \overset{\pm\pm}{\delta}{}^{nn} P'_{TFR}(y) \big) P'_T(y) \tag{2.17}$$
$$- P_T(y) (\Lambda_R - \lambda)^{-1} \big(P_{TFR}(y) \Delta(+\infty) \Delta^{-1}(y) \overset{\pm\pm}{\delta}{}^{nn} - \Delta(+\infty) \Delta^{-1}(y) \overset{\pm\pm}{\delta}{}^{nn} P'_{TFR}(y) \big) \Big] \Big] \Big\} .$$

Hence, transformation of the elements of the transition matrix is determined by the following relation:

$$\sum_e \Big\{ \delta_{ie} - \sum_{k\neq n} S_{ik} (S^{-1})_{ke} (1 - B_e) \Big\} S'_{en} = (1 - i I_{nn}) S_{in} \tag{2.18}$$
$$i, n = 1, \dots, N$$

where I_{nn} is given by formula (2.17).

The fairly complex transformation law becomes simple, if the following equalities are fulfilled:

$$I_{nn} = 1 - B_n - (S^{-1}(1-B)S')_{nn} \qquad n=1,\dots,N \tag{2.19}$$

i.e.

$$C = B \tag{2.20}$$

In this case,

$$S \to S' = B^{-1} S B . \tag{2.21}$$

It should be mentioned that the diagonal elements of the matrix S are invariant under the transformations (2.21). In the general case of Eq.(2.3)

$$S_{nn} \to S'_{nn} = \frac{C_n}{B_n} S_{nn} \qquad (n=1,\dots,N) \tag{2.22}$$

The set of all transformations of the form (2.14) is given by the set of all diagonal matrices $B(\lambda)$. Therefore, just as the set of diagonal matrices, transformations (2.14) (conserving the spectral problem (1.1)) form the infinite-dimensional Abelian Lie group B the "parameters" of which are arbitrary functions $B_e(\lambda)$ $(e=1,\dots,N)$.

III. The general form of integrable equations and Backlund transformations

1. The infinite-dimensional group B of transformations (2.14), under which the spectral problem (1.1) is invariant, contains the transformations of various types. Let us examine the one-parametric subgroup of the group B, which is given by the matrix

$$B(\lambda) = \exp(-i(t'-t)Y(\lambda)) \tag{3.1}$$

where $Y(\lambda)$ is an arbitrary diagonal matrix (and $C=B$). As It's easy to see, this group is a group of time-displaycements:

$$S(\lambda,t) \to S'(\lambda,t) = \exp\{i(t'-t)Y(\lambda)\} S(\lambda,t) \exp\{-i(t'-t)Y(\lambda)\} = S(\lambda,t'). \tag{3.2}$$

Inversion of the mapping $P(x,t) \to S(\lambda,t)$ induces the corresponding transformation $P(x,t) \to P'(x,t) = P(x,t')$. It is of the form

$$\sum_{k,e}\{\exp(-i(t'-t)Y_k(\Lambda_e^+))\}_{mnke} P_{ke}(x,t')\Delta_k - \sum_{k,e}\{\exp(-i(t'-t)Y_e(\Lambda_e^+))\}_{mnke} P_{ke}(x,t)\Delta_e = 0 . \tag{3.3}$$

Operators Λ_e^+ are given by formula (2.13), in which one should put $P'(x,t) = P(x,t')$. For the case $N = 2$, the relations of such a type were found in Refs./7,8/.

Formula (3.3) determines inexplicitly the evolution of $P(x,t)$ in time t: $P(x,t) \to P(x,t')$. Let us consider the infinitesimal displaycement $t \to t' = t + \varepsilon$, $\varepsilon \to 0$

$$P(x,t) \to P(x,t+\varepsilon) = P(x,t) + \varepsilon \frac{\partial P(x,t)}{\partial t} .$$

Then, from Eq.(3.3) we obtain partial differential equations

$$\frac{\partial P_{mn}(x,t)}{\partial t} + i\left[P_F(x,t), \int_{-\infty}^{x} dy \frac{\partial P_D(y,t)}{\partial t}\right]_{mn} + i\sum_{k,e}\{Y_e(L_e^+) - Y_k(L_e^+)\}_{mnke} P_{ke} = 0 \tag{3.4}$$

$$(m \neq n, \quad m,n = 1,\dots,N)$$

where $L^+ = \Lambda^+(P'=P)$, i.e.

$$L^+ \cdot = -i\frac{\partial}{\partial x} - \left[P(x), \cdot\right]_F - i\left[P_F(x), \int_{-\infty}^{x} dy \left[P_F(y), \cdot\right]_D\right] \qquad (3.5)$$

Correspondingly,

$$\frac{dS(\lambda,t)}{dt} = i\left[Y(\lambda), S(\lambda,t)\right]. \qquad (3.6)$$

Partial differential equations (3.4) are just the equations integrable by the inverse scattering method with the help of the linear spectral problem (1.1). Using the IST method equations (Gelfand-Levitan-Marchenko equations), one can find a broad class of solutions of Eqs.(3.4) (multi-soliton solutions). At $N = 2$ and $P_D = 0$ we have equations studied in Ref./6/. Some concrete equations of the (3.4) type with $N \geq 3$ are well known. The model of resonantly interacting wave envelops /19-21/ correspond to linear functions $Y_e(\lambda)$ ($Y_e(\lambda) = Y_e \cdot \lambda$, $e = 1, \ldots, N$) and the multi-component nonlinear Schrödinger equation /22/ - to quadratic functions $Y_e(\lambda)$ ($Y_e(\lambda) = Y_e \cdot \lambda^2$). Equations of the type (3.4) at $Y_e(\lambda) = Y_e \cdot \lambda^{-1}$ will be examined in the next section.

A broader class (than Eq.(3.4) of integrable equations appears, if P (as in the case $N = 2$ /7/) depends, in addition to t , on a few variables $\vec{y}$ of time type. Examining the t - and $\vec{y}$ ($\delta P = \varepsilon\left(\frac{\partial P}{\partial t} + \vec{H}(\lambda, t, \vec{y}) \cdot \frac{\partial P}{\partial \vec{y}}\right)$)- infinitesimal displaycement, we get from Eq.(2.14):

$$\frac{\partial P_{mn}(x,t,\vec{y})}{\partial t} + \sum_{k,e}\left\{\vec{H}(L^+_R, t, \vec{y})\right\}_{mnke} \frac{\partial P_{ke}}{\partial \vec{y}} + i\left\{\left[P_F(x,t,\vec{y}), \int_{-\infty}^{x} dz \frac{\partial P_D(z,t,\vec{y})}{\partial t}\right] + \vec{H}(L^+_R, t, \vec{y})\left[P_F(x,t,\vec{y}), \int_{-\infty}^{x} dz \frac{\partial P_D(z,t,\vec{y})}{\partial \vec{y}}\right]\right\}_{mn} + i\sum_{k,e}\left\{Y_e(L^+_R, t, \vec{y}) - Y_k(L^+_R, t, \vec{y})\right\}_{mnke} P_{ke} = 0 \qquad (3.7)$$

$$m \neq n, \quad m, n = 1, \ldots, N$$

Thus, the class of equations integrable by means of the linear spectral problem (1.1) is characterized by the integro-differential operator L^+_R and $N+n-3$ arbitrary functions $\vec{H}(\lambda, t, \vec{y})$, $Y_e(\lambda, t, \vec{y}) - Y_k(\lambda, t, \vec{y})$ ($e, k = 1, \ldots, N$) (n is the number of independent variables).

In the particular case,

$$Y(\lambda) = \Omega(\lambda)\, Y ,$$

where Y is the constant diagonal matrix, $\Omega(\lambda)$ an arbitrary function, Eqs.(3.4) may be written out in the compact form:

$$\frac{\partial P_F}{\partial t} + i\left[P_F, \Gamma\right]_F - i\Omega_F(L^+_R)\left[Y, P_F\right] = 0 \qquad (3.8)$$

$$\Gamma = \int_{-\infty}^{x} dz \frac{\partial P_D(z,t)}{\partial t}$$

or

$$\frac{\partial P}{\partial t} - \frac{\partial P_D}{\partial t} + i\left[P, \Gamma\right] - i\Omega_F(L^+_R)\left[Y, P\right] = 0 . \qquad (3.9)$$

2. Let us now attract our attention to the fact that, by virtue of Eq.(3.2) (or (3.6), the diagonal elements of the transition matrix are time-independent:

$$\frac{dS_D(\lambda)}{dt} = 0 \qquad (3.10)$$

Hence, $S_{nn}(\lambda)$ ($n = 1, \ldots, N$) are the generating functionals of the integrals of motion. Expanding $\ln S_D(\lambda)$, as usual, in a series of λ^{-1}

$$\ln S_D(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} C^{(n)} , \qquad (3.11)$$

we obtain the infinite series of the integrals of motion $\{C^{(n)}, n = 0, 1, 2, \ldots, \infty\}$ ($C^{(n)}$ are the diagonal matrices with elements $C^{(n)}_m$, $m = 1, \ldots, N$). Expressions for. $C^{(n)}$ in terms of $P(x,t)$ may be found through the use of the procedure proposed in Ref./19/. Let us present here its somewhat modified form.

Let us represent the fundamental matrix F^+ as follows:

$$F^+(x,\lambda)=R(x,\lambda)\,E(x,\lambda)\,\exp\Big\{\int_x^{\infty}dy\,\chi(y,\lambda)\Big\} \qquad (3.12)$$

where E is the asymptotic of the linear problem (1.1), $\chi(y,\lambda)$ the diagonal matrix, and the matrix R satisfies the condition $R_{\mathcal{D}}=1$. From Eq.(3.12) we have

$$\ln S_{\mathcal{D}}(\lambda)=\int_{-\infty}^{+\infty}dy\,\chi(y,\lambda) \qquad (3.13)$$

Substituting Eq.(3.12) into Eq.(1.1), we find

$$\frac{\partial R}{\partial x}+i\lambda[R,A]-R\chi-iPR=0 \qquad (3.14)$$

Expanding χ and R in asymptotic series of λ^{-1}

$$\chi(x,\lambda)=\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\chi^{(n)}(x), \qquad (3.15)$$
$$R(x,\lambda)=1+\sum_{n=1}^{\infty}\frac{1}{\lambda^n}R^{(n)}(x)$$

we obtain the recurrent relations

$$\frac{\partial R^{(n)}}{\partial x}-i[A,R^{(n+1)}]-\chi^{(n)}-\sum_{p=1}^{n}R^{(p)}\chi^{(n-p)}-iPR^{(n)}=0$$
$$n=1,2,\dots, \qquad (3.16)$$
$$-i[A,R^{(1)}]=iP+\chi^{(0)}.$$

It follows from Eq.(3.16) that

$$\chi^{(0)}=-iP_{\mathcal{D}},\quad \chi^{(n)}=-i(PR_F^{(n)})_{\mathcal{D}}\quad n=1,2,\dots, \qquad (3.17)$$

and $R_F^{(n)}$ are determined from the recurrent relations

$$\frac{\partial R_F^{(n)}}{\partial x}-i[A,R_F^{(n+1)}]+i\sum_{p=1}^{n-1}R_F^{(p)}(PR_F^{(n-p)})_{\mathcal{D}}-i(PR_F^{(n)})_F+iR_F^{(n)}P_{\mathcal{D}}=0 \qquad (3.18)$$
$$n=1,2,\dots,$$

$$R_F^{(1)}=-P_{FR},$$

Formulae (3.17) and (3.18) enable us to calculate all integrals of motion, which, by virtue of Eqs.(3.11) and (3.15), are

$$C^{(n)}=\int_{-\infty}^{+\infty}dx\,\chi^{(n)}(x) \qquad (n=0,1,2,\dots) \qquad (3.19)$$

Note, that for all equations of the class (3.4) the integrals of motion $C^{(n)}$ are of the same form, with an accuracy of the concrete reductions of P.

3. Each concrete equation of the type (3.4) is characterized by a definite matrix $Y(\lambda)$ and, correspondingly, by a definite form of the time dependence (3.2) of the transition matrix. It's easy to prove that transformations (2.14) with matrix B, which is independent of t and $\vec{y}$, conserve the form of the time dependence of the matrix S. Hence, they transform the solutions of an equation of the type (3.4) into the solutions of the same equation, i.e. these are usual (auto-) Backlund transformations. The group of Backlund transformations contains the group of transformations (2.21) as a subgroup. These transformations do not change the diagonal elements of the transition matrix (and hence, the Hamiltonian) and, therefore, form an infinite-dimensional group of symmetry. It may be shown that the integrals of motion (3.19) are connected just to these groups of symmetry.

We shall refer the transformations (2.14), which change $S_{nn}(\lambda)$ (i.e. $C\neq B$), to as Backlund transformations. Similarly to the case $N = 2$ /23/, the infinite Abelian group of Backlund transformations is a direct product $B_c\otimes B_d$ of the infinite-dimensional continuous group B_c of continual Backlund transformations and the infinite discrete group B_d of soliton Backlund transformations. The group B_c includes transformations not changing the number of zeros in the diagonal elements of the transition matrix. Soliton Backlund transformations are the transformations (2.3) changing the number of zeros in $S_{nn}(\lambda)$ and, hence, adding one, or several solitons to the initial solution. The structure and properties of Backlund transformations at $N \geq 3$ will be considered in considerable detail elsewhere.

Transformations (2.14) with matrix B, which is dependent of t and (or) $\vec{y}$, are the generalized Backlund transformations (for the case $N = 2$ see Ref./7/): they change the form of the time dependence of the matrix S, thereby converting into each other the solutions of different (with different Y_e and $\vec{H}$) equations of the type (3.7).

Thus, we see that the one-parametric groups of time displayce-ments, which generate the partial differential equations of the type (3.4), the symmetry groups of these equations, the groups of Backlund transformations and generalized Backlund transformations, they all are the subgroups of the infinite group of transformations conserving the spectral problem (1.1).

IV. Integrable equations with singular dispersion law. Relativistically-invariant equations.

The matrix $Y(\lambda)$ coincides, as it's easy to see from Eq.(3.4), with the dispersion matrix of the linearized equation* (as in the case $N = 2$ /6/). For entire functions $Y_e(\lambda)$ the explicit form of integrable equations is found by direct calculation.

* Here and below we shall consider the equations with two independent variables x, t.

In the case of the singular dispersion law (for example, $Y(\lambda) = (\lambda - \lambda_0)^{-n} Y$) we apply the method proposed in Ref./14/.

Let us consider the equation of the form

$$\frac{\partial P_F}{\partial t} + i[P, \Gamma] - i\left(\overset{+}{L}_R - \lambda_0\right)_F^{-n}[Y, P] = 0 \tag{4.1}$$

where n is an arbitrary integer positive number. In Appendix 1 it is shown that

$$\left(\overset{+}{L}_R - \lambda\right)\left[A, \sum_{m=1}^{N} Y_m \frac{\bar{\phi}_{FT}^{+(mm)}}{S_{mm}}\right] = \left[P_F(x), Y\right]$$

Taking into account that

$$\sum_{m=1}^{N} Y_m \frac{\bar{\phi}_T^{+(mm)}}{S_{mm}} = F^+ Y S_D^{-1} F^{-1} \overset{df}{=} \Pi(x,t,\lambda) \quad , \tag{4.2}$$

we have

$$\left(\overset{+}{L}_R - \lambda\right)^{-1}[Y, P(x,t)] = -\left[A, \Pi_F(x,t,\lambda)\right] \tag{4.3}$$

Hence,

$$\left(\overset{+}{L}_R - \lambda_0\right)^{-n}[Y, P(x,t)] = -\frac{1}{(n-1)!}\left[A, \frac{\partial^{n-1}\Pi_F(x,t,\lambda)}{\partial \lambda^{n-1}}\Big|_{\lambda=\lambda_0}\right]. \tag{4.4}$$

Thus, the equation (4.1) is of the form

$$\frac{\partial P_F}{\partial t} + i[P, \Gamma] + \frac{i}{(n-1)!}\left[A, \frac{\partial^{n-1}\Pi(x,t,\lambda)}{\partial \lambda^{n-1}}\Big|_{\lambda=\lambda_0}\right] = 0. \tag{4.5}$$

By virtue of the singularity of the dispersion law, it is required (for the case of $N = 2$ see Ref./14/) that

$$S_F(\lambda_0) = 0 \, . \tag{4.6}$$

Therefore,

$$\Pi(x,t,\lambda_0) = F^+(x,t,\lambda_0)\, Y\, F^{+-1}(x,t,\lambda_0). \tag{4.7}$$

The quantity $\Pi(x,t,\lambda)$ satisfies an equation which is easily found from the formula (A1.1) and definitions (4.2). It is of the form

$$\frac{\partial \Pi}{\partial x} = i\lambda[A,\Pi] + i[P,\Pi] . \qquad (4.8)$$

Solving Eq.(4.8) with respect to Π and substituting into Eq.(4.5), one can find an equation which is satisfied by $P(x,t)$.

Let us examine the case when $n = 1$ and $\lambda_o = 0$ in more detail. We have (since $[A,\Pi_D] = 0$)

$$\frac{\partial P_F}{\partial t} + i[P,\Gamma] + i[A,\Pi(x,t,0)] = 0 \qquad (4.9)$$

$$\frac{\partial \Pi(x,t,0)}{\partial x} = i[P(x,t),\Pi(x,t,0)] . \qquad (4.10)$$

By virtue of Eq.(4.7),

$$\Pi(x,t,0) = F^+(x,t,0)\, Y\, F^{+\,-1}(x,t,0) \qquad (4.11)$$

and from Eq.(1.1)

$$P(x,t) = i\, F^+(x,t,0)\,\frac{\partial F^{+\,-1}(x,t,0)}{\partial x} . \qquad (4.12)$$

Equation (4.10) is satisfied identically, by virtue of Eqs.(4.11) and (4.12), and Eq.(4.9) is of the form $(F^+ = F^+(x,t,0))$

$$\frac{\partial}{\partial t}\Big(F^+\frac{\partial F^{+\,-1}}{\partial x}\Big)_F + i\Big[F^+\frac{\partial F^{+\,-1}}{\partial x},\Gamma\Big] + \big[A, F^+ Y F^{+\,-1}\big] = 0 . \qquad (4.13)$$

Equation (4.13) is invariant under Lorenz transformations $x \to x' = \rho x$, $t \to t' = \rho^{-1} x$ (x, t are the cone variables). Also, it has the invariant group sense where $F^+ \in$ the local group G, and $i\, F^+\frac{\partial F^{+\,-1}}{\partial x} = P \in$ the algebra of local group G.

At $N = 2$ ($P_D = 0$, $P_{21} = -P_{12}$) Eq.(4.13) is the sine-Gordon

equation /6,14/. At $N \geq 3$ it is the generalisation of the sine-Gordon equation to an arbitrary group G , and it was considered in Refs./15,16,5/ for the first time. For a full accord with Refs./15,16/ one must put $F^{+\,-1}(x,t,0) = U(x,t) \in G$:

$$\frac{\partial}{\partial t}\Big(U^{-1}\frac{\partial U}{\partial x}\Big)_F - \Big[U^{-1}\frac{\partial U}{\partial x}, \int_{-\infty}^{x} dy\, \frac{\partial}{\partial t}\Big(U^{-1}\frac{\partial U}{\partial y}\Big)_D\Big] + \big[A, U^{-1} Y U\big] = 0 \qquad (4.14)$$

In Refs./5/ and /16/ these equations have been shown to be gauge-equivalent to the equations of the principal chiral field equations in a space of G/H (where H is the group of diagonal matrices).

Thus, among the equations integrable by the spectral problem (1.1) there are a broad class of relativistically-invariant equations (4.14). Backlund transformations for these equations are given by relations (2.14) and the conservation laws - by formulae (3.17)-(3.19) (with $P = i\,U^{-1}\frac{\partial U}{\partial x}$).

Equations (4.14), which are a generalisation of the sine-Gordon equation to the general linear group $GL(N)$, allows the natural group reductions:

1) the reduction to the group $SU(N)$:

$$U^+(x,t)\,U(x,t) = 1 , \quad A^+ = A , \quad Y^+ = Y \quad (P^+(x,t) = P(x,t)) \qquad (4.15)$$

2) the reduction to the group $SO(N)$:

$$U_T(x,t)\,U(x,t) = 1 \qquad (P_T(x,t) = -P(x,t)) \qquad (4.16)$$

and iA and iY are the arbitrary real diagonal matrices;

3) the reduction to the group $Sp(N)$ (N is even):

$$U_T(x,t)\, J\, U(x,t) = J , \quad A_T = JAJ , \quad Y_T = JYJ \quad (P_T = JPJ) \qquad (4.17)$$

where $\mathcal{Y}$ is the antisymmetric matrix which may be chosen, for example, in the form $\mathcal{Y}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (1 is the unit matrix of the order of $N/2$).

The reductions (4.15) and (4.16) have been also examined in Refs./15,16/.

The reductions (4.15)-(4.17) also take place for the equations with non-singular matrix $Y(\lambda)$.

At even N it is of interest the case

$$P = i\begin{pmatrix}0 & q\\ 1 & 0\end{pmatrix}, \quad A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.18}$$

where q is the matrix of the order of $N/2$. Under fulfilment of Eq.(4.18) the linear spectral problem (1.1) is equivalent to the system ($\psi=\begin{pmatrix}u\\ v\end{pmatrix}$, u and v have $N/2$ components)

$$-\frac{\partial^2 v}{\partial x^2} + q(x,t)v = \lambda^2 v \tag{4.19}$$

Equations integrable by means of the spectral problem (4.19) have been considered in Ref./8/.

In the general case and in the reductions (4.15, 4.16) the Hamiltonian structure of Eqs.(3.8) will be analysed in the next sections. For the reductions (4.17, 4.18) this analysis will be made in a separate paper.

V. Hamiltonian structure of integrable equations

It should be mentioned, first of all, that Eqs.(3.8) are gauge-equivalent to the equations which do not include the term $[P,\Gamma]$. Indeed, let us make the transformation $\psi\to\tilde\psi = \exp(-i\int_{-\infty}^{x} dy\, P_{\mathcal{D}}(y))\psi$. In this case, the spectral problem (1.1) is transformed into the system

$$\frac{\partial\tilde\psi}{\partial x} = i\lambda A\tilde\psi + i\tilde P\tilde\psi$$

where

$$\tilde P = \exp\Big(-i\int_{-\infty}^{x} dy\, P_{\mathcal{D}}(y)\Big)\, P\, \exp\Big(i\int_{-\infty}^{x} dy\, P_{\mathcal{D}}(y)\Big) - P_{\mathcal{D}} = \exp\Big(-i\int_{-\infty}^{x} dy\, P_{\mathcal{D}}(y)\Big)\, P_F\, \exp\Big(i\int_{-\infty}^{x} dy\, P_{\mathcal{D}}(y)\Big).$$

Equation (3.8) is converted into the equation

$$\frac{\partial\tilde P}{\partial t} - i\,\Omega(\tilde L_R^+)[Y,\tilde P]=0 \tag{5.1}$$

where the operator $\tilde L^+$ is given by formula (3.5), in which one should make the substitution $P\to\tilde P$ and take into account that $\tilde P_F = \tilde P$ ($\tilde P_{\mathcal{D}}=0$) . Of course, the results of the preceding sections and Appendices (with simplification $\tilde P_{\mathcal{D}} = 0$) are true for the equations of the form (5.1) and operator $\tilde L^+$.

We are going now to prove that Eqs.(5.1) (which are gauge-equivalent to the equations (3.8)) are Hamiltonian ones at arbitrary N.*

Let us first consider the case of the general position and reduction $P^+=P$. For the sake of simplicity, for the non-singular dispersion law let us confine ourselves to equations of the form

$$\frac{\partial\tilde P}{\partial t} - i\left(\tilde L_R^+\right)_F^n[Y,\tilde P]=0 \tag{5.2}$$

where n is an arbitrary positive integer number.

From Eqs.(A1.13) and (4.2) we have

$$[A,\tilde\Pi(x,t,\lambda)] = (\lambda-\tilde L_R^+)^{-1}[Y,\tilde P(x,t)] \tag{5.3}$$

---

* The Hamiltonian structure of Eqs.(5.1) at $N = 2$ is analysed in Ref./14/ in considerable detail.

Expanding either parts of Eq.(5.3) in the asymptotic series of λ^{-1}, we find

$$\tilde{L}_R^{+n}[Y, \tilde{P}(x,t)] = [A, \tilde{\Pi}^{(n+1)}(x,t)]$$

where

$$\tilde{\Pi}(x,t,\lambda) \sim \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \tilde{\Pi}^{(n)}(x,t) . \qquad (5.4)$$

On the other hand, it follows from (2.2) that

$$\delta \ln \hat{S}_{mm} = -i \int_{-\infty}^{+\infty} dx \sum_{k,e} \delta \tilde{P}_{ke} \frac{\bar{\Phi}^{+(mm)}_{ke}}{S_{mm}} \quad (m=1,\dots,N) \qquad (5.5)$$

where $\delta \tilde{P}$ is the arbitrary variation of $\hat{P}$.

Hence,

$$\tilde{\Pi}(x,t,\lambda) = i \frac{\delta\, tr (Y \ln \tilde{S}_D(\lambda))}{\delta \tilde{P}_T(x,t)} \qquad (5.6)$$

where $\delta/\delta p$ is the variational derivative and tr denotes the matrix trace. Taking into account Eqs.(5.4) and (3.11), we get

$$\tilde{\Pi}^{(n)}(x,t) = i \frac{\delta\, tr (Y C^{(n)})}{\delta \tilde{P}_T(x,t)} \qquad (5.7)$$

where $C^{(n)}$ is the diagonal matrix of the integrals of motion.

Hence, equation (5.2) may be written as follows:

$$\frac{\partial \tilde{P}}{\partial t} + \left[A, \frac{\delta H_n}{\delta \tilde{P}_T}\right] = 0 \qquad (5.8)$$

where $H_n = tr (Y C^{(n+1)})$. In the case of $P_D = 0$, the analogous result was obtained in Ref./9/.

Equation (5.8) is of the form

$$\frac{\partial \tilde{P}}{\partial t} = \{\tilde{P}, H_n\} \qquad (5.9)$$

if one gives the following Poisson brackets ($I(p)$, $H(p)$ are the scalar functionals)

$$\{I, H\} = \int_{-\infty}^{+\infty} dx\, tr \left(\frac{\delta I}{\delta \tilde{P}} \left[A, \frac{\delta H}{\delta \tilde{P}}\right]\right). \qquad (5.10)$$

The quantities $\tilde{P}$ and $\tilde{P}_{TR}$ form a pair of canonical (matrix) variables. The results obtained here are also valid at $P^+ = P$: the pairs of canonical variables form the quantities located symmetrically in respect to the diagonal - they may be considered as the independent ones.

The Poisson bracket (5.10) is not the only bracket corresponding to Eq.(5.2). Similarly to the case when $N = 2$ /17,18/, the infinite set of symplectic structures is associated with equations of the form (5.1). Let us consider the following Poisson bracket (for N = 2 see Ref./18/):

$$\{I, H\}_m = \int_{-\infty}^{+\infty} dx\, tr \left(\frac{\delta I}{\delta \tilde{P}} \tilde{L}_R^{+m} \left[A, \frac{\delta H}{\delta \tilde{P}}\right]\right). \qquad (5.11)$$

It is easy to see that Eq.(5.2) is of the form

$$\frac{\partial \tilde{P}}{\partial t} = \{\tilde{P}, H_{n-m}\}_m \qquad (5.12)$$

where m is an arbitrary integer number. Hence, the infinite set of Hamiltonian-Poisson bracket pairs correspond to the concrete equation of the form (5.2).

Let us prove the Hamiltonian structure of Eqs.(5.1) with $\Omega = (\lambda - \lambda_0)^{-n}$. From the relation (5.3) we have

$$(\tilde{L}_R^{+} - \lambda_0)^{-n} [Y, \tilde{P}] = \frac{1}{(n-1)!} \left[A, \left.\frac{\partial^{n-1} \tilde{\Pi}(x,t,\lambda)}{\partial \lambda^{n-1}}\right|_{\lambda=\lambda_0}\right] \qquad (5.13)$$

Taking into account Eq.(5.6), we find

$$\frac{\partial \widetilde{P}}{\partial t}+\left[A, \frac{\delta}{\delta \widetilde{P}_T} tr\left(\frac{1}{(n-1)!} Y \frac{\partial^{n-1} \ln \widetilde{S}_{\mathscr{D}}(\lambda)}{\partial \lambda^{n-1}}\bigg|_{\lambda=\lambda_0}\right)\right]=0 \quad (5.14)$$

Thus, equations (5.1) with $\Omega=(\lambda-\lambda_0)^{-n}$, which are gauge-equivalent to equations (4.1), are Hamiltonian ones with the Poisson bracket (5.10) and Hamiltonian

$$H_n=\frac{1}{(n-1)!}\, tr\left(Y \frac{\partial^{n-1} \ln \widetilde{S}_{\mathscr{D}}(\lambda)}{\partial \lambda^{n-1}}\bigg|_{\lambda=\lambda_0}\right). \quad (5.15)$$

In particular, Eqs.(5.1) with $\Omega=\lambda^{-1}$, which are gauge-equivalent to the generalisations of the sine-gordon equation to the groups $GL(N)$ and $SU(N)$ (see the preceding section),* are Hamiltonian equations. The Hamiltonian of these equations is of the form

$$H=tr\left(Y \ln \widetilde{S}_{\mathscr{D}}(0)\right). \quad (5.16)$$

Expression (5.16) may be transformed as follows. From the relation (2.2) we have

$$\widetilde{S}-1=-i\int_{-\infty}^{+\infty} dx\, exp(-i\lambda A x)\, \widetilde{P}(x,t)\, \widehat{F}^{+}(x,t,\lambda). \quad (5.17)$$

Hence,

$$\ln \widetilde{S}_{\mathscr{D}}(0)=\ln\left\{1-\widehat{F}^{+}_{\mathscr{D}}(x=+\infty,t,0)+\widehat{F}^{+}_{\mathscr{D}}(x=-\infty,t,0)\right\} \quad (5.18)$$

---

* The Hamiltonian structure of Eq.(4.14) under the group $SU(2)$ was proved in Ref./5/. In our work N is arbitrary. Note also that the transformation $P\to\widetilde{P}=exp(-iR)P_F\, exp\, iR\ \left(R=\int_{-\infty}^{x} dy\, P_{\mathscr{D}}(y)\right)$, for Eqs.(4.14), which, as we have seen, is the gauge transformation, has been considered in Ref./16/.

Thus, the Hamiltonian (5.16) is

$$H=tr\left\{Y \ln\left(1+\widetilde{\mathcal{U}}^{-1}_{\mathscr{D}}(x=-\infty,t)-\widetilde{\mathcal{U}}^{-1}_{\mathscr{D}}(x=+\infty,t)\right)\right\} \quad (5.19)$$

where $\quad \widetilde{\mathcal{U}}(x,t)=\widehat{F}^{+-1}(x,t,0).$

In the combined cases when $Y(\lambda)$ contains the singular and regular parts, the Hamiltonian structure of the equations is proved in a similar way.

It should be mentioned that the integrals of motion $C^{(n)}$ ($n=1,2,\ldots$) of equations of the form (5.1) are connected to the form-invariance of these equations with respect to the transformations which in the infinitesimal form are the following:

$$\delta_n \widetilde{P}(x,t)=-\varepsilon\left[A, \frac{\delta C^{(n)}}{\delta \widetilde{P}_T(x,t)}\right]=i\varepsilon\left(\widetilde{L}^{+}_{R}\right)^{n-1}_{F}\left[Y, \widetilde{P}(x,t)\right] \quad n=1,2,\ldots,$$

where ε is the diagonal matrix (of the order of N) of parameters.

VI. The structure of equations at $P_T=-P$.

In this case, not all the variables P_{ke} are dynamically independent. For this reason, the analysis of the foregoing sections should be modified.

Let us introduce the upper triangular matrix Q with zeros along the diagonal, such that

$$P=Q-Q_T \quad (6.1)$$

i.e. $\quad Q_{ke}=P_{ke} \quad$ at $\quad e>k$

$\quad\quad Q_{ke}=0 \quad$ at $\quad e\le k$

Now, let us transform Eqs.(3.8) in such a way that they contain only the independent dynamical variables Q.*

To this end, we return to Eqs.(2.7). They may be written at $B=1-i\varepsilon Y$ and $P'=P+\varepsilon\frac{\partial P}{\partial t}$ (see (3.1)-(3.4)) in the following form $(\overset{++}{\Phi}{}^{(in)}_{ke}=(F^{-1}_+)_{ik}(F^+)_{en})$:

$$\int_{-\infty}^{+\infty}dx\, tr\Big\{\Big(\frac{\partial P_T}{\partial t}-i[Y,P]_T\Omega(\lambda)\Big)\overset{++}{\Phi}{}^{(in)}\Big\}=0\,. \quad (i\neq n) \tag{6.2}$$

Substituting the definition (6.1) into Eq.(6.2) and using the properties of the trace, we obtain

$$\int_{-\infty}^{+\infty}dx\, tr\Big\{\frac{\partial Q_T}{\partial t}\big(\overset{++}{\Phi}{}^{(in)}-\overset{++}{\Phi}{}^{(in)}_T\big)-i[Y,Q]_T\Omega(\lambda)\big(\overset{++}{\Phi}{}^{(in)}+\overset{++}{\Phi}{}^{(in)}_T\big)\Big\}=0 \quad (i\neq n) \tag{6.3}$$

Then let us introduce the projection operations Δ_+ and Δ_- :

$$(Z_{\Delta_+})_{ke}=\begin{cases} Z_{ke} & \text{at} \quad e>k \\ 0 & \text{at} \quad e<k, \end{cases}$$

$$(Z_{\Delta_-})_{ke}=\begin{cases} 0 & \text{at} \quad e>k \\ Z_{ke} & \text{at} \quad e<k \end{cases} \tag{6.4}$$

where Z is an arbitrary matrix with zeros along the diagonal. It is clear that $Z=Z_{\Delta_+}+Z_{\Delta_-}$, $Z_{\Delta_\pm\Delta_\pm}=Z_{\Delta_\pm}$. Since $Q_{\Delta_+}=Q$, $Q_{T\Delta_+}=0$, $Q_{T\Delta_-}=Q_T$, Eq.(6.3) is equivalent to the following

$$\int_{-\infty}^{+\infty}dx\, tr\Big\{\frac{\partial Q_T}{\partial t}\overset{++}{\xi}{}^{(in)}_{\Delta_+}-i[Y,Q]_T\Omega(\lambda)\overset{++}{\psi}{}^{(in)}_{\Delta_+}\Big\}=0 \tag{6.5}$$

where $\psi=\varphi_F+\varphi_{FT}$, $\xi=\varphi_F-\varphi_{FT}$.

Equation (6.5) already contains the independent variables only. Hence, the transition from Eq.(6.2) to Eq.(6.5) is the projection onto the subspace of independent dynamical variables.

* The results of this section are true for the more general case of Eqs.(3.4).

The first term of Eq.(6.5) can be converted into the form which contains ψ_{Δ_+} instead of ξ_{Δ_+}. Let us define the quantity $W(x,t)$ by the relation

$$\frac{\partial Q}{\partial t}=\mathcal{D}^+_{\Delta_+}W \tag{6.6}$$

where $\mathcal{D}^+$ is the "covariant" derivative:

$$\mathcal{D}^+\cdot=\frac{\partial}{\partial x}-i[Q-Q_T,\cdot\,]_F+i[Q-Q_T,\cdot\,]_{FT} \tag{6.7}$$

Taking into account that $\frac{\partial Q_T}{\partial t}=\mathcal{D}^+_{\Delta_-}W_T$, integrating by parts and using formula (A2.6), we find (assuming $W(x=-\infty)=0$)

$$\int_{-\infty}^{+\infty}dx\, tr\Big(\frac{\partial Q_T}{\partial t}\overset{++}{\xi}{}^{(in)}_{\Delta_+}\Big)=-\int_{-\infty}^{+\infty}dx\, tr\Big(W_T\mathcal{D}^+_{\Delta_+}\overset{++}{\xi}{}^{(in)}_{\Delta_+}\Big)=$$

$$=i\lambda\int_{-\infty}^{+\infty}dx\, tr\Big(W_T(x,t)[A,\overset{++}{\psi}{}^{(in)}_{\Delta_+}]\Big) \tag{6.8}$$

As a result, Eq.(6.5) is of the form

$$\int_{-\infty}^{+\infty}dx\, tr\Big\{W_T[A,\overset{++}{\psi}{}^{(in)}_{\Delta_+}]-[Y,Q]_T\,\omega(\lambda^2)\,\overset{++}{\psi}{}^{(in)}_{\Delta_+}\Big\}=0 \quad (i\neq n) \tag{6.9}$$

where we put $\Omega(\lambda)=\lambda\,\omega(\lambda^2)$, because at $P_T=-P$ $\Omega(\lambda)$ should be the antisymmetric function λ.

In Appendix II it is shown that the following relation holds in the subspace stretched over $\overset{++}{\psi}{}^{(in)}_{\Delta_+}$:

$$L^{(Q)}_{\Delta_+R}\,\overset{++}{\psi}{}^{(in)}_{\Delta_+}=\lambda^2\,\overset{++}{\psi}{}^{(in)}_{\Delta_+} \quad (i\neq n) \tag{6.10}$$

where

$$L^{(Q)}\psi=-\mathcal{D}^+\mathcal{D}^-_{\Delta_+R}\psi+2\mathcal{D}^+\Big[Q,\int_x^{\infty}dy[Q-Q_T,\psi]_{\mathcal{D}}\Big]_R\,. \tag{6.11}$$

Therefore, Eq.(6.9) may be written as

$$\int_{-\infty}^{+\infty} dx\, tr\left\{W_T[A, \stackrel{++}{\Psi}{}^{(n)}_{\Delta+}] - [Y,Q]_T\,\omega(L^{(Q)}_{\Delta+R})\stackrel{++}{\Psi}{}^{(n)}_{\Delta+}\right\} = 0_{(i\neq n)}. \qquad (6.12)$$

Finally, making the transition from the operator $L^{(Q)}_{\Delta+}$ to the adjoint operator $L^{(Q)+}_{\Delta+}$ and taking into account the equality $tr(W_T[A,\Psi_{\Delta+}]) = tr(\Psi_{T\Delta-}[A,W])$, we obtain

$$\int_{-\infty}^{+\infty} dx\, tr\left\{\stackrel{++}{\Psi}{}^{(n)}_{T\Delta-}\left([A,W] - \omega(L^{(Q)+}_{\Delta+R})[Y,Q]\right)\right\} = 0 \qquad (6.13)$$

where

$$L^{(Q)+}\cdot = -D^{-} D^{+}_{\Delta+R}\cdot - 2\left[Q(x), \int_{-\infty}^{x} dy\,[Q-Q_T, D^{+}_{\Delta+R}\cdot]_{D}\right] \qquad (6.14)$$

Equality (6.13) is fulfilled, if

$$[A,W] - \omega(L^{(Q)+}_{\Delta+R})[Y,Q] = 0.$$

Using (6.6), we obtain the following differential equations $(D \stackrel{\text{H}}{=} D^{+})$:

$$\frac{\partial Q}{\partial t} - D_{\Delta+}\,\omega_R(L^{(Q)+}_{\Delta+R})[Y,Q] = 0. \qquad (6.15)$$

There is no difficulty in seeing that Eq.(6.15), which contains Q only, is equivalent to Eq.(3.8) at $P_T = -P$. At $N=2$ $D = \frac{\partial}{\partial x}$ and Eq.(6.15) coincides with equations considered in Ref./14/.

For singular functions $\omega(\lambda^2)$ of the type

$$\omega(\lambda^2) = (\lambda^2 - \lambda_0^2)^{-n} \qquad (6.16)$$

we use the analog of formula (4.3) which in our case is of the form (see Appendix II)

$$\frac{1}{2}\left[A, \frac{\Pi_Q(x,t,\lambda)}{\lambda}\right] = \left(L^{(Q)+}_{\Delta+R} - \lambda^2\right)^{-1}[Y,Q] \qquad (6.17)$$

where

$$\Pi_Q(x,t,\lambda) = \sum_{m=1}^{N} Y_m \frac{\stackrel{-+}{Y}{}^{(mm)}_{\Delta+}}{S_{mm}}. \qquad (6.18)$$

Hence,

$$\left(L^{(Q)+}_{\Delta+R} - \lambda_0^2\right)^{-n}[Y,Q] = -\frac{1}{2(n-1)!}\left[A, \frac{\partial^{n-1}\,\Pi_Q(x,t,\lambda)/\lambda}{\partial(\lambda^2)^{n-1}}\Big|_{\lambda=\lambda_0}\right] \qquad (6.19)$$

Thus, the equations with the dispersion law (6.16) are of the form

$$\frac{\partial Q}{\partial t} + \frac{1}{2(n-1)!} D_{\Delta+}\left[A, \frac{\partial^{n-1}}{\partial(\lambda^2)^{n-1}}\frac{\Pi_Q(x,t,\lambda)}{\lambda}\Big|_{\lambda=\lambda_0}\right] = 0. \qquad (6.20)$$

VII. The Hamiltonian structure of equations at $P_T = -P$.

Equations of the type (6.15) contain only the dynamical independent quantities and admit the natural Hamiltonian structure.

Let us first consider the equations of the form

$$\frac{\partial Q}{\partial t} - i D_{\Delta+}\left(L^{(Q)+}_{\Delta+R}\right)^{n}[Y,Q] = 0. \qquad (7.1)$$

From Eq.(2.2) (see also (5.5)) we have

$$\delta \ln S_{mm} = -i\int_{-\infty}^{+\infty} dx\, tr\left(Q_{T\Delta-}\frac{\stackrel{-+}{Y}{}^{(mm)}_{\Delta+}}{S_{mm}}\right) \qquad (7.2)$$

Hence,

$$\Pi_Q(x,t,\lambda) = i\frac{\delta}{\delta Q(x,t)}\, tr(Y \ln S_D(\lambda)). \qquad (7.3)$$

Expanding the left- and right-hand parts of Eq.(6.17) in the asymptotic series of λ^{-1}, we find

$$\left(L^{(Q)+}_{\Delta+R}\right)^n [Y,Q] = -\frac{1}{2}\left[A, \Pi_Q^{(2n+1)}\right].$$

Taking into account Eqs.(7.3) and (3.11), we obtain

$$\left(L^{(Q)+}_{\Delta+R}\right)^n [Y,Q] = -\frac{i}{2}\left[A, \frac{\delta}{\delta Q}\, tr\left(Y C^{(2n+1)}\right)\right]. \qquad (7.4)$$

Hence, Eq.(7.1) may be written as follows:

$$\frac{\partial Q}{\partial t} = \frac{i}{2}\, \mathcal{D}_{\Delta+}\left[A, \frac{\delta}{\delta Q}\, tr\left(Y C^{(2n+1)}\right)\right] = 0 \,. \qquad (7.5)$$

It is obvious that Eq.(7.5) is a Hamiltonian one. The Poisson bracket is of the form

$$\{I, H\} = \int_{-\infty}^{+\infty} dx\, tr\left\{\frac{\delta I}{\delta Q_T}\, \mathcal{D}_{\Delta+}\left[A, \frac{\delta H}{\delta Q}\right]\right\} \qquad (7.6)$$

and the Hamiltonian is equal to

$$H_n = \frac{1}{2}\, tr\left(Y C^{(2n+1)}\right). \qquad (7.7)$$

It is clear that for the equation

$$\frac{\partial Q}{\partial t} - i\, \mathcal{D}_{\Delta+}\, \omega\left(L^{(Q)+}_{\Delta+R}\right)[Y, Q] = 0 \qquad (7.8)$$

where $\omega(\lambda^2) = \sum_\alpha \omega_\alpha (\lambda^2)^\alpha$, the Hamiltonian H_ω is equal to

$$H_\omega = \frac{1}{2}\, tr\left(Y \sum_\alpha \omega_\alpha C^{(2\alpha+1)}\right). \qquad (7.9)$$

Just as in the case of the general situation (see section V), the infinite number of Hamiltonian-Poisson bracket pairs is connected to the equations of the form (7.8).

Let us proceed now to the equations with the singular dispersion law. Let us prove the Hamiltonian structure of the equation

$$\frac{\partial Q}{\partial t} - \mathcal{D}_{\Delta+}\left(L^{(Q)+}_{\Delta+R}\right)^{-1} [Y, Q] = 0 \qquad (7.10)$$

The case $\omega = (\lambda^2)^{-n}$ is analysed in a similar way. From Eq. 6.17) we have that

$$\frac{\partial Q}{\partial t} - \frac{i}{2}\, \mathcal{D}_{\Delta+}\left[A, \frac{\Pi_Q(x,t,\lambda)}{\lambda}\Big|_{\lambda=0}\right]_R = 0 \,. \qquad (7.11)$$

By virtue of Eq.(7.3),

$$\frac{\Pi_Q(x,t,\lambda)}{\lambda}\Big|_{\lambda=0} = \left\{\frac{\partial}{\partial\lambda}\, \Pi_Q(x,t,\lambda)\right\}\Big|_{\lambda=0} = i\frac{\delta}{\delta Q}\, tr\left(Y\left\{\frac{\partial}{\partial\lambda}\ln S_{\mathcal{D}}(\lambda)\right\}\Big|_{\lambda=0}\right) \qquad (7.12)$$

Hence, Eq.(7.10) is of the form

$$\frac{\partial Q}{\partial t} = \{Q, H\} \qquad (7.13)$$

where

$$H = \frac{i}{2}\, tr\left(Y \frac{\partial}{\partial\lambda}\ln S_{\mathcal{D}}(0)\right) \qquad (7.14)$$

and the Poisson bracket is given by formula (7.6).

It's easy to prove that Eq.(7.10) is equivalent to the $SO(N)$-Gordon equation (the sine-Gordon equation under the group $SO(N)$) which has been considered in section IV. Indeed, taking into account (by virtue of $S_F(\lambda=0) = 0$) that

$$\Pi_Q = \sum_{m=1}^{N} Y_m \int^{+(mm)}_{\Delta+}$$

and using (a2.4), we find

$$\frac{\partial Q}{\partial t} + \frac{i}{2}\left[A, \Pi_{T\Delta+} + \Pi_{\Delta+}\right] = 0 \qquad (7.15)$$

where Π is given by formula (4.7). Transposing (7.15) and substracting the resulting equation from (7.15), we derive Eq.(4.9) where $P = Q - Q_T$.

Thus, we have shown that the $SO(N)$-Gordon equation at arbitrary N is a Hamiltonian one with the Poisson bracket (7.6). Let us reduce the Hamiltonian (7.14) of this equation to a more explicit form. For this purpose, we use the identity

$$iF^{-1}(x,t,\lambda)\,A\,F^{+}(x,t,\lambda) - iAS = \frac{\partial}{\partial x}\left(-\frac{\partial F^{-1}(x,t,\lambda)}{\partial\lambda}F^{+}(x,t,\lambda)\right) - iAS \qquad (7.16)$$

which is obtained directly from (1.1).

We find from Eq.(7.16) that

$$\frac{\partial}{\partial\lambda}\ln S_{mm} = -i\int_{-\infty}^{+\infty}dx\left\{(F^{-1}AF^{+})_{mm}\frac{1}{S_{mm}} - A_{mm}\right\} \qquad (7.17)$$

$$(m = 1,\dots,N)$$

Hence (taking into account that $S_F(\lambda=0)=0$),

$$H = \frac{i}{2}\sum_{m=1}^{N} Y_m \frac{\partial}{\partial\lambda}\ln S_{mm}(0) =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}dx\,\mathrm{tr}\left(AF^{+}(x,t,0)YF^{+-1}(x,t,0) - AY\right). \qquad (7.18)$$

Denoting $F^{+-1}(x,t,0) = \mathcal{U}(x,t)$ (see section IV), we have

$$H = \frac{1}{2}\int_{-\infty}^{+\infty}dx\,\mathrm{tr}\left(A\mathcal{U}_T Y\mathcal{U} - AY\right), \qquad (7.19)$$

In this notation

$$Q = i\left(\mathcal{U}_T\frac{\partial\mathcal{U}}{\partial x}\right)_{A_+} \qquad (7.20)$$

As we have already noted, the Hamiltonian structure of the $SO(3)$-Gordon equation has been proved in Refs./15,16/. The Hamiltonian in those papers coincides (with an accuracy of the transition to

cone variables) with the Hamiltonian (7.19). However, the canonical variables are different. In Refs./15/ and /16/ the canonical variables are natural coordinates of the local group $SO(3)$. In our case (as it is seen from Eq.(7.20)), the natural coordinates in the local algebra $SO(N)$ play the role of canonical coordinates. The situation for the $GL(N)$-Gordon and $SU(N)$-Gordon equations is similar (see section V).

Note, that by virtue of the gauge equivalence of the G-Gordon equations to the equations of the principal chiral field over the space of flags /16,5/ the latter are Hamiltonian ones.

In conclusion, in should be mentioned that the Langangian structure of some equations connected to the principal chiral field and namely, the four-fermion type equations, has been proved in the work /24/.

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APPENDIX I

In this Appendix we shall obtain some relations which include the quantities $\tilde{\Phi}^{(in)}_{ke} = (\Psi^{-1})_{ik}(\Psi')_{en}$ and integro-differential operators $\Lambda, \Lambda^{+}, L, L^{+}$.

We shall use the following notation: $\overset{+\pm}{\tilde{\Phi}}{}^{(in)}_{ke} = (F^{+^{-1}})_{ik}(F^{+})_{en}$, $\overset{\mp\pm}{\tilde{\Phi}}{}^{(in)}_{ke} = (F^{-^{-1}})_{ik}(F^{+})_{en}$. From Eq.(1.1) we find

$$\frac{\partial \tilde{\Phi}^{(in)}}{\partial x} = -i\lambda\left[A, \tilde{\Phi}^{(in)}\right] + i\,\tilde{\Phi}^{(in)} P'_T - i P_T \tilde{\Phi}^{(in)} . \qquad \text{(A1.1)}$$

Hence,

$$\frac{\partial \tilde{\Phi}^{(in)}_{\mathcal{D}}}{\partial x} = i(P'_{\mathcal{D}} - P_{\mathcal{D}})\tilde{\Phi}^{(in)}_{\mathcal{D}} + i\left(\tilde{\Phi}^{(in)}_F P'_T - P_T \tilde{\Phi}^{(in)}_F\right)_{\mathcal{D}} . \qquad \text{(A1.2)}$$

From Eq.(A1.2) we have

$$\tilde{\Phi}^{(in)}_{\mathcal{D}}(x) = \Delta(+\infty)\,\Delta^{-1}(x)\,\tilde{\Phi}^{(in)}_{\mathcal{D}}(+\infty) - i\,\Delta^{-1}(x)\int_x^{\infty} dy\, \Delta(y)\left(\tilde{\Phi}^{(in)}_F(y) P'_T(y) - P_T(y)\tilde{\Phi}^{(in)}_F(y)\right)_{\mathcal{D}} \qquad \text{(A1.3)}$$

or

$$\tilde{\Phi}^{(in)}_{\mathcal{D}}(x) = \Delta(+\infty)\,\Delta^{-1}(x)\,\tilde{\Phi}^{(in)}_{\mathcal{D}}(-\infty) + i\,\Delta^{-1}(x)\int_{-\infty}^{x} dy\, \Delta(y)\left(\tilde{\Phi}^{(in)}_F(y) P'_T(y) - P_T(y)\tilde{\Phi}^{(in)}_F(y)\right)_{\mathcal{D}} \qquad \text{(A1.4)}$$

where $\Delta(x) = \exp\{ i\int_{-\infty}^{x} dy\,(P_{\mathcal{D}}(y) - P'_{\mathcal{D}}(y))\}$.

Following from the asymptotic properties of F^{+} and F^{-}, we get

$$\left(\overset{+\pm}{\Phi}{}^{(in)}_{\mathcal{D}}(+\infty)\right)_{kk} \overset{df}{=} \left(\overset{+\pm}{\delta}{}^{in}\right)_{kk} = \delta_{ik}\,\delta_{kn},$$
$$\left(\overset{\mp}{\Phi}{}^{(in)}_{\mathcal{D}}(-\infty)\right)_{kk} \overset{df}{=} \left(\overset{-\pm}{\delta}{}^{in}\right)_{kk} = \delta_{ik}\,S_{kn} . \qquad \text{(A1.5)}$$

From Eqs.(A1.1), (A1.3), and (A1.5) we obtain

$$\Lambda\,\overset{+\pm}{\Phi}{}^{(in)}_F = \lambda\left[A, \overset{+\pm}{\Phi}{}^{(in)}_F\right] + P_{TF}(x)\,\Delta(+\infty)\,\Delta^{-1}(x)\,\overset{+\pm}{\delta}{}^{in} - \Delta(+\infty)\,\Delta^{-1}(x)\,\overset{+\pm}{\delta}{}^{in} P'_{TF}(x) \qquad \text{(A1.6)}$$

where

$$\Lambda\Phi = i\frac{\partial\Phi}{\partial x} + \left(\Phi P'_T(x) - P_T(x)\Phi\right)_F - i\,\Delta^{-1}(x)\int_x^{\infty} dy\,\Delta(y)\left(\Phi(y)P'_T(y) - P_T(y)\Phi(y)\right)_{\mathcal{D}} P'_{TF}(x) + i\,P_{TF}(x)\,\Delta^{-1}(x)\int_x^{\infty} dy\,\Delta(y)\left(\Phi(y)P'_T(y) - P_T(y)\Phi(y)\right)_{\mathcal{D}} . \qquad \text{(A1.7)}$$

In particular,

$$\Lambda\,\overset{+\pm}{\Phi}{}^{(in)}_F = \lambda\left[A, \overset{+\pm}{\Phi}{}^{(in)}_F\right] \qquad (i \neq n) \qquad \text{(A1.8)}$$

i.e.

$$\Lambda_R\,\overset{+\pm}{\Phi}{}^{(in)}_F = \lambda\,\overset{+\pm}{\Phi}{}^{(in)}_F \qquad (i \neq n) \qquad \text{(A1.9)}$$

There is no difficulty in seeing that for the operator

$$L^{+}\cdot \overset{df}{=} \Lambda^{+}(P' = P) = -i\frac{\partial}{\partial x} - \left[P(x), \cdot\,\right]_F - i\left[P_F(x), \int_{-\infty}^{x} dy\left[P_F(y), \cdot\,\right]_{\mathcal{D}}\right] \qquad \text{(A1.10)}$$

the following relation holds $\left(\overset{-\pm}{\Phi}{}^{(in)}_{ke} = (F^{-^{-1}})_{ik}(F^{+})_{en}\right)$

$$L^{+}\,\overset{-\pm}{\Phi}{}^{(in)}_{TF} = \lambda\left[A, \overset{-\pm}{\Phi}{}^{(in)}_{TF}\right] + \left[P_F(x), \overset{-\pm}{\delta}{}^{in}\right] . \qquad \text{(A1.11)}$$

Hence,

$$L^{+}\,\frac{\overset{-\pm}{\Phi}{}^{(nn)}_{TF}}{S_{nn}} = \lambda\left[A, \frac{\overset{-\pm}{\Phi}{}^{(nn)}_{TF}}{S_{nn}}\right] + \left[P_F(x), \overset{-\pm}{\delta}{}^{nn}\right] \qquad \text{(A1.12)}$$

where $(\overset{-+}{\delta}{}^{nn})_{kk} = \delta_{kn}$.

Multiplying the equality (A1.12) by Y_n and summing over n, we obtain

$$\overset{+}{L} \sum_{n=1}^{N} Y_n \frac{\overset{-+}{\varphi}{}^{(nn)}_{TF}}{S_{nn}} = \lambda\Big[A, \sum_{n=1}^{N} Y_n \frac{\overset{-+}{\varphi}{}^{(nn)}_{TF}}{S_{nn}}\Big] + \big[P_F(x), Y\big], \qquad \text{(A1.13)}$$

APPENDIX II

Let us consider Eq.(A1.6) at $P' = P$ and $P_T = -P = -(Q - Q_T)$. We use the following notation: $\overset{\pm+}{\varphi}{}^{(in)}_{kl} = (F^{\pm^{-1}})_{ik}(F^{+})_{ln}$ and $\psi = \varphi_F + \varphi_{FT}$, $\chi = \varphi_F - \varphi_{FT}$. It's easy to prove that (for $\varphi = \overset{++}{\varphi}$)

$$i\frac{\partial \psi}{\partial x} + [Q - Q_T, \psi]_F - i\Big[Q - Q_T, \int_x^{\infty} dy\,[Q - Q_T, \psi]_{\mathcal{D}}\Big] = \\ = \lambda[A, \chi] - 2[Q - Q_T, \overset{++}{\delta}], \qquad \text{(A2.1)}$$

$$i\frac{\partial \chi}{\partial x} + [Q - Q_T, \chi]_F = \lambda[A, \psi]. \qquad \text{(A2.2)}$$

Let us apply the operation Δ_+ to Eqs.(A2.1) and (A2.2). We have the following result:

$$i\frac{\partial \psi_{\Delta_+}}{\partial x} + [Q - Q_T, \psi_{\Delta_+}]_{F\Delta_+} + [Q - Q_T, \psi_{\Delta_+}]_{TF\Delta_+} \\ - 2i\Big[Q, \int_x^{\infty} dy\,[Q - Q_T, \psi_{\Delta_+}]_{\mathcal{D}}\Big] = \lambda[A, \chi_{\Delta_+}] - 2[Q, \overset{++}{\delta}], \qquad \text{(A2.3)}$$

$$i\frac{\partial \chi_{\Delta_+}}{\partial x} + [Q - Q_T, \chi_{\Delta_+}]_{F\Delta_+} - [Q - Q_T, \chi_{\Delta_+}]_{TF\Delta_+} = \lambda[A, \psi_{\Delta_+}]. \qquad \text{(A2.4)}$$

In expressions of the type $Z_{TF\Delta_+\dots}$ the operations are performed from left to right. Note also some obvious but useful properties in calculations: operations T, F, $\mathcal{D}$ commute with each other; $T\Delta_+ = \Delta_- T$; $[P_{\Delta_\pm}, Z_{\Delta_\pm}]_{\Delta_\mp} = 0$; for the symmetric matrix φ we have $\varphi = \varphi_{\Delta_+} + \varphi_{\Delta_+ T}$, for the antisymmetric one χ,

$\chi = \chi_{\Delta_+} - \chi_{\Delta_+ T}$ and so on.

Let us introduce the "covariant" derivatives $\mathcal{D}^-$, $\mathcal{D}^+$:

$$\mathcal{D}^{\pm} = \frac{\partial}{\partial x} - i[Q - Q_T, \cdot\,]_F \pm i[Q - Q_T, \cdot\,]_{TF} \qquad \text{(A2.5)}$$

Then Eq.(A2.4) is of the form

$$i\mathcal{D}^{+}_{\Delta_+}\chi_{\Delta_+} = \lambda[A, \psi_{\Delta_+}]. \qquad \text{(A2.6)}$$

Applying the operation R to Eq.(A2.3), acting on the resulting equation $\mathcal{D}^{+}_{\Delta_+}$ and taking into account (A2.6), we find

$$L^{(Q)}_{\Delta_+}\overset{++}{\psi}{}^{(in)}_{\Delta_+} = \lambda^2[A, \overset{++}{\psi}{}^{(in)}_{\Delta_+}] - 2i\mathcal{D}^{+}_{\Delta_+}[Q_R, \overset{++}{\delta}{}^{in}] \qquad \text{(A2.7)}$$

where

$$L^{(Q)}\psi = \mathcal{D}^{+}\Big\{-\mathcal{D}^{-}_{\Delta_+ R}\psi + 2\Big[Q(x), \int_x^{\infty} dy\,[Q - Q_T, \psi(y)]_{\mathcal{D}}\Big]_R\Big\}. \qquad \text{(A2.8)}$$

Hence,

$$L^{(Q)}_{\Delta_+ R}\overset{++}{\psi}{}^{(in)}_{\Delta_+} = \lambda^2\overset{++}{\psi}{}^{(in)}_{\Delta_+} \quad (i \neq n). \qquad \text{(A2.9)}$$

Equations for $\overset{-+}{\psi}_{\Delta_+}$ and $\overset{-+}{\chi}_{\Delta_+}$ are the following:

$$i\mathcal{D}^{-}_{\Delta_+}\overset{-+}{\psi}{}^{(in)}_{\Delta_+} + 2i\Big[Q(x), \int_{-\infty}^{x} dy\,\big[Q(y) - Q_T(y), \overset{-+}{\psi}{}^{(in)}_{\Delta_+}(y)\big]_{\mathcal{D}}\Big] = \\ = \lambda[A, \overset{-+}{\chi}{}^{(in)}_{\Delta_+}] - 2[Q, \overset{-+}{\delta}{}^{in}], \qquad \text{(A2.10)}$$

$$i\mathcal{D}^{+}_{\Delta_+}\overset{-+}{\chi}{}^{(in)}_{\Delta_+} = \lambda[A, \overset{-+}{\psi}{}^{(in)}_{\Delta_+}]. \qquad \text{(A2.11)}$$

With the use of Eqs.(A2.10) and (A2.11) we obtain

$$-\mathcal{D}^{-}_{\Delta_+}\mathcal{D}^{+}_{\Delta_+ R}\overset{-+}{\chi}{}^{(in)}_{\Delta_+} - 2\Big[Q(x), \int_{-\infty}^{x} dy\,[Q - Q_T, \mathcal{D}^{+}_{\Delta_+ R}\overset{-+}{\chi}{}^{(in)}_{\Delta_+}]_{\mathcal{D}}\Big] = \\ = \lambda^2[A, \overset{-+}{\chi}{}^{(in)}_{\Delta_+}] - 2\lambda[Q, \overset{-+}{\delta}{}^{in}]. \qquad \text{(A2.12)}$$

Comparison of the left-hand side of Eq.(A2.12) and Eq.(6.14) gives

$$\mathcal{L}^{(Q)+}_{\Delta_+} \bar{\Upsilon}^{+(in)}_{\Delta_+} = \lambda^2 \left[A, \bar{\Upsilon}^{+(in)}_{\Delta_+}\right] - 2\lambda \left[Q, \bar{\delta}^{+in}\right]. \qquad (A2.13)$$

Hence,

$$\left(\mathcal{L}^{(Q)+}_{\Delta_+ R} - \lambda^2\right)\left[A, \Pi_Q(x,t,\lambda)\right] = 2\lambda\left[Y, Q\right]. \qquad (A2.14)$$

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