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V.V.Mazepus

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V.V.Mazepus

Institute of Nuclear Physics,
630090, Novosibirsk, USSR

A b s t r a c t

A simple microscopic model of nuclear rotation is considered. The particle-plus-rotor model is shown to give the exact solution of the problem, while the cranking model disagrees with it. The comparison with the approximate projection method is made.

TESTING OF SOME NUCLEAR ROTATION MODELS IN A SIMPLE MICROSCOPIC SYSTEM

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Institute of Nuclear Physics,
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Presently there exist various approach to the problem of micriscopic description of nuclear rotation. Some of them have been developed in order to improve the cranking model (CM), or to find a rogirous base for it /1,2/, whereas the others are based on an approximate solution of the exact operator equations of motion /3/ and their results are close to those of the corresponding versions of the Bohr-Mottelson unified model (the particle-plus-rotor model, the phonons-rotation coupling model etc.).

For even nuclei with small angular momenta the consequencies of these models are in fairly good agreement but for odd nuclei the results obtained differ sharply even for low-lying states. This discrepancy is most obvious in the case of strong Coriolis mixing of rotational bands. For example, in ^{159}Dy the CM agrees much better with experimental data compared to the particle-plus-rotor model (PRM) /4/.

So far neither model has no rigorous base. The PRM seems to be more preferable due to its clearness. It is easily derived from the operator equations of motion if the odd particle is considered to be only connected with the rotational degree of freedom of the even core. The CM takes into account exactly the Pauli principle but because of the semi-classical character its

applicability within small angular momenta is still open to discussion. Several years ago an attempt of its substantiation has been made in the framework of the approximate projection method (APM) /2/. This attempt has proved to be insufficient as, first of all, the APM itself must be substantiated. For these reasons, it seems to be useful to study the applicability of the above methods in simple models of nuclear rotation which admit an exact or regular approximate solution and are comparable with realistic Hamiltonians, at least to some extent.

In the present paper we shall consider a simple model wherein splitting of one degenerated "spherical" level produces a large number of levels due to deformation and an one-particle operator modelling the angular momentum (plane rotation) is conserved. We shall show that in more important, from the physical point of view, variant of the odd system the PRM gives the exact solution of our problem, while the CM disagrees with it (except some trivial cases). Moreover, we shall see that the APM leads to the PRM equations in contradiction to the general results of Ref./2/. The inaccuracy made in the above paper is also analysed.

DESCRIPTION OF THE MODEL

The space of single-particle states in our model is determined by the basis $|\ell_z s_z \nu\rangle$, $\ell_z = -\ell, \dots, +\ell$; $s_z = \pm\frac{1}{2}$; $\nu = 1, \dots 2\Omega$. We shall consider ℓ_z and s_z as the z-components of the formal angular momentum operators in the representation $\vec{\ell}^2 = \ell(\ell+1)$, $\vec{s}^2 = \frac{3}{4}$; ν is an additional quantum number over which all the operators in the problem are degenerated. The Hamiltonian in the model is expressed through additive operators $\vec{L} = \sum_{12} (\vec{\ell})_{12} a_1^+ a_2$ and $\vec{S} = \sum_{12} (\vec{s})_{12} a_1^+ a_2$ (a_1^+ is the fermion-creation operator in

the single-particle state $|1\rangle$):

$$H = -\frac{\varkappa}{2}\{S_+ S_-\} - \gamma(L_+ S_- + L_- S_+) = \\ = -\varkappa \vec{S}^2 + \varkappa S_z^2 - \gamma(L_+ S_- + L_- S_+), \qquad (1)$$

where $L_\pm = L_x \pm i L_y$, $S_\pm = S_x \pm i S_y$, $\{AB\} = AB + BA$.

In what follows ℓ is assumed to be integer. Hence, our model can be interpreted as a version of the spin-orbital coupling on the only shell.

Hamiltonian (1) conserves the operator $S_z + L_z \equiv M$, which plays a role of the angular momentum in this model. M takes on integer values in the even-particle-number system and semi-integer ones in the odd system. The maximum possible particle number in the configuration space of the model is $2\cdot(2\ell+1)\cdot 2\Omega$; we shall consider the case when a half of the states is only occupied in the even system and a half plus one state in the odd one.

The conditions

$$\varkappa > 0, \quad \gamma > 0, \quad \Omega \gg 1, \quad \varkappa \sim \gamma \ell \Omega, \qquad (2)$$

are considered to be fulfilled. In addition, we shall be interested in not too large values of the momentum:

$$|M| \ll (2\ell+1)\Omega. \qquad (3)$$

EXACT SOLUTION

Since the additive operators $\vec{L}$ and $\vec{S}$ satisfy just the same commutation rules as the single-particle matrices $\vec{\ell}$ and $\vec{s}$, the internal states of Hamiltonian (1) form the series, or bands, corresponding to the representations of $SU_2 \times SU_2$ algebra with certain "total angular momenta" L and S. The problem on admissible values for L and S is analogous to that on electron terms in an atom. If we are only interested in the yrast-states,

we should consider the representations with as large S as possible since the Hamiltonian includes $\vec{S}^2 = S(S+1)$ with the large negative coefficient. For the even system, as mentioned above, we suppose that a half of all possible states is occupied, i.e. the particle number is $N = 2\Omega(2l+1)$; the representation with a maximum S is then given by quantum numbers $L=0$, $S=\Omega(2l+1)$. For the odd system $N=2\Omega(2l+1)\pm 1$ and the corresponding representation is characterized by the values $L=l$, $S=\Omega(2l+1)-\frac{1}{2}$. To prove rigorously that just these representations correspond to the yrast-band is a complicated problem. However, the numerical experiment has shown that when $\Omega \leq 10$ and $l = 1$, the intersection of the bands corresponding to the close representations is possible only at the momenta comparable with the extreme one, i.e. at $M \sim \Omega(2l+1)$. As we are most interested in the case of the small momenta, our further considerations will be concerned with the representations with a maximum S.

For the even system the solution is trivial since in this case $L=0$, $M=S_z$ and consequently,

$$H = \varkappa M^2 + \text{const}, \qquad (4)$$

that corresponds to a plane rotator with the inertial parameter

$$\mathcal{J} = \frac{1}{2\varkappa} . \qquad (5)$$

We shall find below that the Inglis inertial parameter calculated in this model yields the same value.

In the case of the odd system, L_z is restricted (in the representation where L_z is diagonal) by the inequalities

$$-l \leq L_z \leq l \qquad \text{and} \qquad -S \leq M - L_z \leq S ,$$

because $S_z + L_z = M$; at $|M| \leq (2\Omega-1)(l+\frac{1}{2})$ the second inequality being resulted from the first. Let the quantity M be fixed.

Then the basis states are only characterized by the quantum number L_z, and $\langle L_z|L_+S_-|L'_z\rangle = \delta_{L'_z,L_z-1}\sqrt{(l+L_z)(l+1-L_z)(S-M+L_z)(S+1+M-L_z)} \simeq$

$$\simeq \sqrt{S^2-M^2}\,\langle L_z|L_+|L'_z\rangle ,$$

since $S \gg l$. In the region of interest $M \ll S \simeq \Omega(2l+1)$ and

$$\langle L_z|L_+S_-|L'_z\rangle \simeq S\langle L_z|L_+|L'_z\rangle .$$

Thus, the ground band in the odd system is described by a Hamiltonian

$$H \simeq \varkappa(M-l_z)^2 - 2\gamma S l_x , \qquad (6)$$

where, since $L=l$, we substitute the operators $\vec{L}$ by the single-particle matrices $\vec{l}$ and also omit the additive constant. Note that the sign of the second term in the right-hand part of eq.(6) is not of importance as it can be changed by the unitary transformation $\exp(i\pi l_z)$. Hamiltonian (6) is a matrix of the order of $2l+1$; the eigenvalues of this matrix, which are minimum at a given M, determine the energies of yrast-states.

SELF-CONSISTENT FIELD

We beging a study of various approximate methods for analysingour model with calculations of a self-consistent field and a single-particle density matrix in the frame of the Hartree-Fock method. The initial Hamiltonian (1) may be written down in the standard form

$$H = \sum_{12}\varepsilon_{12}a_1^+a_2 + \frac{1}{2}\sum_{1221'}\langle 12|v|2'1'\rangle a_1^+a_2^+a_{2'}a_{1'} , \qquad (7)$$

where $\varepsilon_{12} = -\gamma(l_+S_- + l_-S_+)_{12}$, $\langle 12|v|2'1'\rangle = -\varkappa\left[(S_+)_{11'}(S_-)_{22'} + (S_-)_{11'}(S_+)_{22'}\right] -$

$$-\gamma\left[(l_+)_{11'}(S_-)_{22'} + (l_+)_{22'}(S_-)_{11'} + (l_-)_{11'}(S_+)_{22'} + (l_-)_{22'}(S_+)_{11'}\right] ,$$

and the matrix elements are taken over any single-particle basis.

The self-consistent field W is determined by a single-particle density matrix as

$$W_{11'} = \varepsilon_{11'} + \langle 12|v|2'4'\rangle \rho_{2'2} - \langle 12|v|1'2'\rangle \rho_{2'2} =$$
$$= -2\left[\varkappa\langle S_\pm\rangle + \gamma\langle l_\pm\rangle\right](s_x)_{11'} - 2\gamma\langle S_\pm\rangle(l_x)_{11'} - \tag{8}$$
$$- \gamma(l_+\rho S_- + l_-\rho S_+)_{11'} + \varkappa(S_+\rho S_- + S_-\rho S_+)_{11'} + \gamma(l_+\rho S_- + l_-\rho S_+ + \text{conj.})_{11'},$$

where we use the notations $\langle S_\pm\rangle = \mathrm{Tr}\, s_\pm \rho$ and $\langle l_\pm\rangle = \mathrm{Tr}\, l_\pm \rho$. The condition $\Omega \gg 1$ enables one to neglect the single-particle and exchange terms in eq.(8); then

$$W = -2\left[\varkappa\langle S_\pm\rangle + \gamma\langle l_\pm\rangle\right]s_x - 2\gamma\langle S_\pm\rangle l_x \,. \tag{9}$$

Hence, W is diagonal in the representation with definite l_x and s_x. Conditions (2) which are assumed to be satisfied lead to that the single-particle energies are decomposed into two series, each of them is characterized by the quantum number $s_x = \pm\frac{1}{2}$. The gap between these series is much wider than that between the levels inside both series. A state with definite l_x and s_x is degenerated over $\nu = 1, \ldots, 2\Omega$. In the even system the states of series $s_x = -\frac{1}{2}$ are occupied; thus,

$$\rho = \tfrac{1}{2} - s_x \,. \tag{10}$$

Whence,

$$\langle l_\pm\rangle = 0\,, \quad \langle S_\pm\rangle = -\Omega(2l+1) \equiv -S \,. \tag{11}$$

Finally,

$$W = 2\varkappa S s_x + 2\gamma S l_x \,. \tag{12}$$

Let us also find the Inglis moment of inertia. It may be written as follows

$$\mathcal{J}_0 = \mathrm{Tr}\, m\Phi \,,$$

where $m = l_z + s_z$ and Φ satisfies the equation

$$[W\Phi] = [m\rho] \,. \tag{13}$$

Then

$$\Phi = \frac{1}{2\varkappa S} s_z \quad \text{and} \quad \mathcal{J}_0 = \frac{1}{2\varkappa} \,,$$

that coincides with eq.(5).

PARTICLE-PLUS-ROTOR MODEL

Consider now the odd system with the particle number

$$N = 2\Omega(2l+1) + 1 \,.$$

The PRM Hamiltonian has the form

$$\mathcal{H} = \frac{(\mathcal{M}-m)^2}{2\mathcal{J}_0} + W = \frac{(\mathcal{M}-l_z-s_z)^2}{2\mathcal{J}_0} + 2\varkappa S s_x + 2\gamma S l_x \,, \tag{14}$$

where $\mathcal{M}$ is the total angular momentum. Only those internal states in eq.(14) corresponding to a finding of an odd particle at the unoccupied level have a physical sense. Due to conditions (2) and (3) this is equivalent to averaging of eq.(14) over a state with $s_x = \frac{1}{2}$. Hence,

$$\mathcal{H}_{PRM} = \frac{(\mathcal{M}-l_z)^2}{2\mathcal{J}_0} + 2\gamma S l_x = \varkappa(\mathcal{M}-l_z)^2 + 2\gamma S l_x \,, \tag{15}$$

that, up to the trivial rotation, coincides with eq.(6). Thus, we have found that the PRM reproduces the exact solution of our model problem if the conditions (2) and (3) are fulfilled.

CRANKING MODEL

The alternative approach to the problem of finding the energies of yarst-states in the odd system arises from the CM. If one introduces a density matrix $\bar{\rho}$ and a self-consistent field $\bar{W}$, both are dependent on rotation frequency ω, and also an energy $\mathcal{E}$ of the system, one can write the equations of this

model as follows:

$$[\bar{W}\bar{\rho}]=0\ ,\quad \bar{\rho}^2=\bar{\rho}\ ,\quad \mathrm{Tr}\,m\bar{\rho}=\mathrm{Tr}(S_z+l_z)\bar{\rho}=\mathcal{M}\ ,\qquad \mathcal{E}=\langle H\rangle\ , \tag{16}$$

where

$$\bar{W}=-2\left[\varkappa\langle S_\pm\rangle+\gamma\langle L_\pm\rangle\right]S_x-2\gamma\langle S_\pm\rangle l_x+\varkappa(S_+\bar{\rho}S_-+S_-\bar{\rho}S_+)-$$
$$-\omega l_z-\omega S_z\ ,\qquad \langle S_\pm\rangle=\mathrm{Tr}\,S_\pm\bar{\rho}\ ,\quad \langle L_\pm\rangle=\mathrm{Tr}\,l_\pm\bar{\rho}\ . \tag{17}$$

$\bar{W}$ includes only those of exchange terms which are of the order $\sim\varkappa$, i.e. the terms of rotational energy. Generally, the terms of such a sort should be taken into an average value of the Hamiltonian:

$$\mathcal{E}=\langle H\rangle=-\varkappa\langle S_\pm\rangle^2-2\gamma\langle S_\pm\rangle\langle L_\pm\rangle+\varkappa\,\mathrm{Tr}\,S_+\bar{\rho}S_-\bar{\rho}\ . \tag{18}$$

An exact solution of the system (16) which corresponds to the occupation of $2\Omega(2l+1)+1$ lower-energy states is readily found by rotation around the y-axis in the l - and S -spaces:

$$\bar{\rho}=\tfrac{1}{2}(1+\delta_\mu g)-(S_x\cos\chi-S_z\sin\chi)(1-\delta_\mu g)\ ,$$
$$g=e^{-i\beta l_y}\,\delta(l_x-\Lambda)\,e^{i\beta l_y}\ , \tag{19}$$

where δ_μ is the matrix defined by

$$\langle\nu|\delta_\mu|\nu'\rangle=\delta_{\nu\nu'}\delta_{\nu\mu}\quad(\nu\text{-space}),$$

the quantum numbers μ and Λ fix the even-particle state and the angles β and χ are defined by

$$\sin\beta=\frac{\omega}{\sqrt{4\gamma^2\langle S_\pm\rangle^2+\omega^2}};\quad 2\left[\varkappa|\langle S_\pm\rangle|-\gamma|\langle L_\pm\rangle|\right]\sin\chi-\omega\cos\chi=\varkappa\sin\chi\ . \tag{20}$$

At small $\mathcal{M}$ a rotational frequency ω is of the order $\varkappa\mathcal{M}$, while $\langle S_\pm\rangle\sim S\sim l\Omega$. It follows that $\omega\ll\varkappa\langle S_\pm\rangle\sim\varkappa S$. Taking into account this condition, we find

$$\langle S_\pm\rangle=\mathrm{Tr}\,S_\pm\bar{\rho}\simeq-S+\frac{\omega^2}{8\varkappa^2 S}\ ,\quad \langle L_\pm\rangle=\mathrm{Tr}\,l_\pm\bar{\rho}=\Lambda\cos\beta\ . \tag{21}$$

Then eq.(18) gives the energy as a function of ω :

$$\mathcal{E}=\frac{\omega^2}{4\varkappa}+\frac{4\gamma^2S^2\Lambda}{\sqrt{4\gamma^2S^2+\omega^2}}+O\left(\frac{\omega^2}{\varkappa S}\right). \tag{22}$$

Finally, from the momentum consistency condition we find a connection between $\mathcal{M}$ and ω :

$$\mathcal{M}=\mathrm{Tr}\,m\bar{\rho}=S\cdot\sin\chi-\Lambda\sin\beta=\frac{\omega}{2\varkappa}-\frac{\Lambda\omega}{\sqrt{4\gamma^2S^2+\omega^2}}+O\left(\frac{\omega^2}{\varkappa^2S^2}\right). \tag{23}$$

In the frame of the CM equations (22) and (23) determine the energies of the odd system at a given momentum $\mathcal{M}$. It is easy to see that the yrast-line corresponds to a choice $\Lambda=-l$ that corresponds to a rotational band constructed on the lower-energy state of the odd particle. Hence,

$$\mathcal{E}=\frac{\omega^2}{4\varkappa}-\frac{4\gamma^2S^2l}{\sqrt{4\gamma^2S^2+\omega^2}}\ ,$$
$$\mathcal{M}=\frac{\omega}{2\varkappa}+\frac{l\omega}{\sqrt{4\gamma^2S^2+\omega^2}}\ ,\qquad S\simeq\Omega(2l+1)\ . \tag{24}$$

Note that the bands constructed on the states $\Lambda>0$ behave in an "anomalous" way at small ω if

$$\varkappa>\frac{\gamma S}{\Lambda}\ .$$

In this case,

$$\frac{\partial\mathcal{M}}{\partial\omega}<0\ ,\quad \frac{\partial\mathcal{E}}{\partial\omega}<0\ ,$$

when ω is less than some critical value.

How is the GM solution (24) related to the "exact" Hamiltonian (6),or (15)? To answer this question, it is necessary to note that eqs.(24) may be derived if one makes the substitution in (6)

$$l_z^2\to 2\langle l_z\rangle l_z-\langle l_z\rangle^2, \tag{25}$$

where the average is taken over the internal state of the Hamiltonian which is a result of this ,

$$\mathcal{H}=\varkappa\left[\mathcal{M}^2-2(\mathcal{M}-\langle l_z\rangle)l_z-\langle l_z\rangle^2\right]+2\gamma S\,l_x\ . \tag{26}$$

Indeed, if one denotes

$$\omega = 2\varkappa(\mathcal{M} - \langle \ell_z \rangle),$$

one obtains from eq.(26)

$$\mathcal{H}_{CM} = \omega\mathcal{M} - \frac{\omega^2}{4\varkappa} + \sqrt{4\gamma^2 S^2 + \omega^2}\, e^{-i\beta\ell_y} \ell_x e^{i\beta\ell_y}, \qquad (27)$$

where the quantity β is determined by eq.(20). After rotation by the angle β one finds immediately:

$$\mathcal{E} = \langle \mathcal{H} \rangle = \omega\mathcal{M} - \frac{\omega^2}{4\varkappa} - \ell\sqrt{4\gamma^2 S^2 + \omega^2} = \frac{\omega^2}{4\varkappa} - \frac{4\gamma^2 S^2 \ell}{\sqrt{4\gamma^2 S^2 + \omega^2}},$$

$$\mathcal{M} = \frac{\omega}{2\varkappa} + \langle \ell_z \rangle = \frac{\omega}{2\varkappa} + \ell \sin\beta = \frac{\omega}{2\varkappa} + \frac{\ell\omega}{\sqrt{4\gamma^2 S^2 + \omega^2}},$$

This coincides with eq.(24). This result is not unlooked-for if one takes into account that, in a matter of fact, the replacement of (25) means a neglect of the quantum-mechanical interference and, therefore, is a variant of the quasi-classical approximation. There is no difficulty to determine the availability of this approximation. Hamiltonian (6) may be written as follows:

$$\mathcal{H} = \mathcal{H}_{CM} + \varkappa(\ell_z - \langle \ell_z \rangle)^2. \qquad (28)$$

If the second term in the right-hand part of eq.(28) is considered as a perturbation and the trivial case $\gamma = 0$ is excepted, it is easy to find the conditions under which the eq.(24) holds. First, the contribution of this perturbation is small if ω is large enough:

$$\omega^2 \gg \varkappa \min(\varkappa, \gamma S\sqrt{\ell}),$$

or if $\ell \sim 1$, $\varkappa \sim \gamma S$,

$$|\omega| \gg \varkappa \quad , \text{ i.e. } \quad |\mathcal{M}| \gg 1 . \qquad (29)$$

Second, approximation (25) is valid for all $\mathcal{M}$ if

$$f \equiv \frac{\varkappa\ell}{2\gamma S} \lesssim 1 \quad \text{and} \quad \varkappa \ll \gamma S. \qquad (30)$$

In the last case, the quantity f serves as a parameter of perturbation theory. It is clear that conditions (30) mean (at $\ell \sim 1$) a smallness of the Coriolis term in (6), or (15). From (24) it follows that

$$\mathcal{E} \simeq \varkappa\mathcal{M}^2 + \text{const}. \qquad (31)$$

Thus, at $\ell \sim 1$ the CM yields a true result only in that case when the Coriolis term in the Hamiltonian of the PRM is small compared to the one-particle energy (conditions (30)), or compared to the rotation energy of the even core (conditions (29)). At $\ell \gg 1$ the applicability of the CM is extended but, as before, the energy is square-law with respect to the momentum (at not too large $\mathcal{M}$):

$$\mathcal{E} \simeq \frac{\mathcal{M}^2}{2\mathcal{J}}, \quad \mathcal{J} = \frac{1}{2\varkappa} + \frac{\ell}{2\gamma S}, \qquad (32)$$

$$\text{if } 1 \ll \ell \lesssim \frac{2\gamma S}{\varkappa}, \quad \varkappa \ll \gamma S \text{ and } \mathcal{M} \ll \ell.$$

PROJECTION METHOD

In the projection method /1/ a class of variational functions consists of the states

$$P_{\mathcal{M}} |\Phi\rangle, \qquad (33)$$

where $P_{\mathcal{M}}$ is the projection operator of the state with momentum $\mathcal{M}$ and $|\Phi\rangle$ is restricted to wavefunctions of the Hartree-Fock type (in the absence of pair forces). In our model the projection method leads to a correct result both for the even and odd systems. This is clear from the fact that the Hartree-Fock states in either cases belong to the spaces of representations of $SU_2 \times SU_2$ algebra, which correspond to a ground state of the Hamiltonian (1). Actually, in the case of the even system, as seen from (10), all the states with $S_x = -\frac{1}{2}$ are occupied and those with $S_x = \frac{1}{2}$ are free.

Hence, the ground (Hartree-Fock) state $|0\rangle$ satisfies the equations

$$\vec{L}|0\rangle = 0\,, \quad S_x|0\rangle = -\Omega(2l+1)|0\rangle\,,$$
$$(S_y - iS_z)|0\rangle = 0\,. \tag{34}$$

Whence,

$$\vec{L}^2|0\rangle = 0\,, \quad \vec{S}^2|0\rangle = S(S+1)|0\rangle\,,$$
$$S = \Omega(2l+1)\,. \tag{35}$$

Let us now consider a state

$$P_M|0\rangle\,. \tag{36}$$

It is obvious that this state belongs to the same representation and is an internal state of the momentum operator, and, thus, of the Hamiltonian here depending only on the momentum.

In the case of the odd system, let us consider a state

$$|\lambda\rangle = a^+_{\lambda\nu\frac{1}{2}}|0\rangle\,,$$

where $a^+_{\lambda m\sigma}$ is the fermion-creation operator in the state with $l_x = m$, $S_x = \sigma$ (of course, the choice $\nu = 1$ is insignificant). Using the commutation a^+ with $\vec{L}$ and $\vec{S}$, it is not difficult to obtain

$$S_x|\lambda\rangle = -\left[\Omega(2l+1) - \tfrac{1}{2}\right]|\lambda\rangle\,, \quad (S_y - iS_z)|\lambda\rangle = 0\,,$$

and also

$$L_x|\lambda\rangle = \lambda|\lambda\rangle\,, \quad (L_y - iL_z)|\lambda\rangle \sim |\lambda - 1\rangle\,, \quad (L_y - iL_z)|l\rangle = 0\,.$$

It follows that $|\lambda\rangle$ is the basis vector of the representation with $L = l$ and $S = \Omega(2l+1) - \frac{1}{2}$:

$$|\lambda\rangle = |L_x = \lambda, L = l; S_x = -S, S = \Omega(2l+1) - \tfrac{1}{2}\rangle\,. \tag{37}$$

Let us show that the basis of this representation can be given as a linear superposition of vectors

$$|M\lambda\rangle = P_M|\lambda\rangle\,.$$

Let

$$|L_z = m\rangle = \sum_\lambda a_\lambda(m)|L_x = \lambda\rangle\;;$$

then,

$$|L_z S_z\rangle \sim P_{S_z}|L_z, S_x = -S\rangle = \sum_{L'_z S'_z} \delta(L_z + S_z - L'_z - S'_z)\delta(L_z - L'_z)P_{S'_z}|L_z, S_x = -S\rangle =$$

$$= \sum_{\lambda L'_z S'_z} a_\lambda(L_z)\delta(L_z + S_z \equiv L'_z - S'_z)P_{L'_z}P_{S'_z}|\lambda\rangle = \sum_{\lambda M L'_z S'_z} a_\lambda(L_z)\delta(L_z + S_z - M)\delta(L'_z + S'_z - M)P_{L'_z}P_{S'_z}|\lambda\rangle =$$

$$= \sum_{\lambda M} a_\lambda(L_z)\delta(L_z + S_z - M)\,|M\lambda\rangle\,.$$

Therefore, the eigenvector of the Hamiltonian (1) when it corresponds both to the representation $L = l$, $S = \Omega(2l+1) - \frac{1}{2}$ and momentum M, can be taken in the form

$$|M\rangle = \sum_\lambda c_\lambda|M\lambda\rangle = P_M\sum_\lambda c_\lambda|\lambda\rangle = P_M\sum_\lambda c_\lambda a^+_{\lambda\nu\frac{1}{2}}|0\rangle \equiv P_M|\Phi\rangle\,. \tag{38}$$

Thus, the mean energy variated over the (33)-type states has a minimum coinciding with the eigenvalue of Hamiltonian (1). However, approximate formulae for projected energy are usually employed in real calculations. These formulae are valid if the momentum dispersion in the ground Hartree-Fock state is large enough. Proceed now to an analysis of the availability of this approximation in our model.

APPROXIMATE PROJECTION

In Ref./2/ the approximate expression for projected energy has been derived. In our case of a plane rotation this expression may be written as follows:

$$\mathcal{E}_M = \frac{\langle\Phi|\hat{H}\hat{P}_M|\Phi\rangle}{\langle\Phi|\hat{P}_M|\Phi\rangle} \approx \langle\hat{H}\rangle - \frac{1}{2}\frac{\langle(\hat{H} - \langle\hat{H}\rangle)(\hat{M} - \langle\hat{M}\rangle)^2\rangle}{\langle(\hat{M} - \langle\hat{M}\rangle)^2\rangle} +$$
$$+ \frac{\langle\hat{H}(\hat{M} - \langle\hat{M}\rangle)\rangle}{\langle(\hat{M} - \langle\hat{M}\rangle)^2\rangle}(M - \langle\hat{M}\rangle) + \frac{1}{2}\frac{\langle(\hat{H} - \langle\hat{H}\rangle)(\hat{M} - \langle\hat{M}\rangle)^2\rangle}{\langle(\hat{M} - \langle\hat{M}\rangle)^2\rangle^2}\left(M - \langle\hat{M}\rangle\right)^2, \tag{39}$$

where all the averages are taken over the state $|\Phi\rangle$.

Let us choose $P_M|\Phi\rangle$ for the odd system in the form of (38) and will variate eq.(39) over parameters C^*_λ. Since the normalization $|\Phi\rangle$ in (39) is arbitrary, we should require that

$$\langle\Phi|\Phi\rangle = \sum_\lambda C^*_\lambda C_\lambda = 1 \,. \tag{40}$$

Calculation of the averages entering eq.(39) is trivial and yields:

$$\begin{aligned}
&\langle\hat{M}\rangle = C^*_\lambda (\ell_z)_{\lambda\lambda'} C_{\lambda'} \,, \quad \langle\hat{M}^2\rangle = \tfrac{1}{2}\mathcal{S} + C^*_\lambda (\ell_z^2)_{\lambda\lambda'} C_{\lambda'} \,, \\
&\langle\hat{H}\rangle = \tfrac{1}{2}\varkappa\mathcal{S} + 2\gamma\mathcal{S}\, C^*_\lambda (\ell_x)_{\lambda\lambda'} C_{\lambda'} \,, \quad \langle\hat{H}\hat{M}\rangle \simeq \tfrac{1}{2}\varkappa\mathcal{S}\, C^*_\lambda (\ell_z)_{\lambda\lambda'} C_{\lambda'} \,, \\
&\langle\hat{H}\hat{M}^2\rangle \simeq \tfrac{1}{4}\varkappa\mathcal{S}(3\mathcal{S}-1) + \tfrac{\varkappa}{2}\mathcal{S}\, C^*_\lambda (\ell_z^2)_{\lambda\lambda'} C_{\lambda'} + \gamma\mathcal{S}^2 C^*_\lambda (\ell_x)_{\lambda\lambda'} C_{\lambda'} \,,
\end{aligned} \tag{41}$$

where $\mathcal{S} = \Omega(2\ell+1) - \frac{1}{2}$, $(\hat{O})_{\lambda\lambda'} \equiv \langle \ell_x = \lambda | \hat{O} | \ell_x = \lambda' \rangle$.

After substitution of (41) into (39) one gets:

$$\mathcal{E}_M = C^*_\lambda \mathcal{H}_{\lambda\lambda'} C_{\lambda'} \,, \quad \mathcal{H} = \varkappa(M - \ell_z)^2 + 2\gamma\mathcal{S}\,\ell_x \,. \tag{42}$$

This coincides with eqs.(6) and (15).

Thus, in our model the APM gives a true answer coinciding with the result of the PRM. A natural question arises how this is related to the general conclusion (see Ref./2/) that the CM always results from eq.(39). The answer is very simple and consists in the following: in the above mentioned paper the second term in the right-hand part of eq.(39) which renormalizes a mean value of the Hamiltonian was omitted with no good grounds. Indeed, in the case of neglecting this term we obtain, instead of (42), the following:

$$\mathcal{E}_M = 2\gamma\mathcal{S}\, C^*_\lambda (\ell_x)_{\lambda\lambda'} C_{\lambda'} + \varkappa(M - C^*_\lambda (\ell_z)_{\lambda\lambda'} C_{\lambda'})^2 \,.$$

And after variation over C^*_λ with the additional condition (40) we find:

$$\mathcal{E} C_\lambda = \overline{\mathcal{H}}_{\lambda\lambda'} C_{\lambda'} \,, \quad \overline{\mathcal{H}} = 2\gamma\mathcal{S}\,\ell_x + \varkappa\left[M^2 - 2M\ell_z + 2\langle\ell_z\rangle\ell_z - \langle\ell_z^2\rangle\right] \,,$$

$$\langle\ell_z\rangle \equiv C^*_\lambda (\ell_z)_{\lambda\lambda'} C_{\lambda'} \,.$$

As we have shown above (see formula (26) etc.), it leads to the CM equations.

CONCLUSION

The comparative analysis of different approximate methods for solving the nuclear rotation problem studied here in the framework of the simple model confirms the applicability of such rigorous, in the sense of quantum character of rotation, approximations as the particle plus rotor one and also both the accurate and approximate projections. Meanwhile, comparison of the exact solution with the semiclassical cranking model gives grounds for serious doubts as to applicability of this very popular method to the system with the odd particle number. From this viewpoint, a quite good agreement between the numerical calculations carried out in the frame of the CM and the experiment /4/ seems to be hardly explainable. Of interest is a further study of the approximate methods based upon much more "rich" models including the pair interaction and containing, at least, two integrals of motion (momentum and particle number).

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