55

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A b s t r a c t

The Yang-Mills potentials are considered which give rise to the same field strengths but do not reduce to each other by a gauge transformation. For a field class, the explicit expressions are obtained for non-unique potentials, their properties are studied.

I. It is well known that in electrodynamics the potentials giving rise to the same expression for a field strength are necessarily related by some gauge transformation. Generally speaking, for non-Abelian gauge theories the situation is somewhat different since there are potentials not related by a gauge transformation but leading to the equal quantity of the field strength in the entire space. Wu and Yang were evidently the first who considered the example of such a kind[1]. Not long after several analogous examples have been supplied[2-5] and the necessary condition of the existence of such ambiguities has been derived[6,7].

In this paper it is studied a class of fields wherein each potential A^a_μ corresponds to such a potential-double B^a_μ that the equality $F^a_{\mu\nu}(A) = F^a_{\mu\nu}(B)$ is fulfilled but there is no gauge transformation relating A^a_μ and B^a_μ. For the fields under consideration one succeeds in finding the explicit form of the potentials-doubles.

The paper covers the following:

In paragragh 2 the equation is derived which is satisfied by potentials-doubles, and its solution is found. Then, it is proved that the potentials obtained are not reducible to each other by a gauge transformation.

In paragraph 3 the asymptotic behavior of the potentials-doubles is studied.

In paragraph 4 it is proved that the non-unique potentials A^a_i and B^a_i can satisfy the non-homogeneous Yang-Mills equations only with different external sources. In particular, in the class of fields in question no potentials-doubles simultaneously satisfying the homogeneous Yang-Mills equations exist.

And finally, note that all the calculations are carried out for the gauge group $SU(2)$.

2. The general form of the "spherical-symmetrical" potentials can be represented as follows:

$$A_i^a(h_1,h_2,h_3)=\frac{1}{er}\left\{(\delta^{ai}-n^a n^i)h_1(r)+n^a n^i h_3(r)+\varepsilon^{iac}n^c h_2(r)\right\}. \qquad (1)$$

Here $n_i = x^i/r$, $r=(x^2)^{1/2}$, and h_1, h_2, h_3 are the dimensionless arbitrary functions of r. The field strengths expressed through h_1, h_2, h_3 are written in the following form:

$$H_k^a=\frac{1}{2}\varepsilon_{ijk}F_{ij}^a=\frac{1}{2}\varepsilon_{ijk}\left[-\partial_i A_j^a+\partial_j A_i^a+2e\varepsilon^{abc}A_i^b A_j^c\right]=$$

$$=\frac{1}{er^2}\left\{(\delta^{ak}-n^a n^k)H_1+n^a n^k H_3+\varepsilon^{kac}n^c H_2\right\}, \qquad (2)$$

$$H_1=2h_1h_3-rh_2'; \quad H_2=2h_2h_3-h_3+h_1'r; \quad H_3=2(h_1^2+h_2^2-h_2).$$

The touch denotes differentiation with respect to r.

Then let us write down the general form analogous to expression (1) for the potential-double $B_i^a(g_1, g_2, g_3)$, by using other functions g_1, g_2, g_3. Our task is to express g_1, g_2, g_3 via h_1, h_2, h_3, from the equality condition $F_{ij}^a(A)=F_{ij}^a(B)$. Hence, we obtain the following set of differential equations for the functions g_1, g_2, g_3:

$$2g_1g_3-rg_2'=2h_1h_3-rh_2' \qquad \text{(a)}$$

$$2g_2g_3-g_3+g_1'r=2h_2h_3-h_3+h_1'r \qquad \text{(b)} \qquad (3)$$

$$g_1^2+g_2^2-g_2=h_1^2+h_2^2-h_2 \qquad \text{(c)}$$

The obvious solution of this system is $g_\alpha = h_\alpha$, here $\alpha = 1,2,3$. However, as it will be shown below, this set of differential equations can be reduced to that of algebraic equations and has the solution differing from the trivial. To find this solution, let us try to exclude g_3 from eqs.(3a) and (3b). In this case the derivatives of the unknown functions g_2' and g_1' are involved only in the combinations $2g_1g_1'+2g_2g_2'-g_2'$ but this expression is exactly equal to the known quantity $2h_1h_1'+2h_2h_2'-h_2'$, what follows from eq.(3c). Thus, all the derivatives of unknown functions are absent in the result, and we obtain g_1 as a function of g_2, h_α, h_α'. After that, substituting this expression for g_1 in eq.(3c), we get the algebraic quadratic equation for g_2:

$$\left[\frac{h_1h_3(2g_1-1)+rh_1'h_1+rh_2'(h_2-g_2)}{rh_1'+2h_2h_3-h_3}\right]^2-g_2+g_2^2-(h_1^2+h_2^2-h_2)=0. \qquad (4)$$

It follows from eq.(4) that we have two roots one of which $g_2=h_2$. The second, nontrivial root can be found if one divides the left-hand side of eq.(4) by (g_2-h_2). The result has the fairly compact form:

$$g_2=1-h_2-\frac{1}{2}\frac{H_1}{H_1^2+H_2^2}\cdot r\frac{dH_3}{dr}, \qquad (5)$$

where H_1, H_2, H_3 are the same as those in eq.(2). When g_2 is found, g_1 and g_3 are expressed in a trivial fashion from eq.(3).

Now there is necessity to prove that $A_i^a(h_\alpha)$ and $B_i^a(g_\alpha)$ do not related by a gauge transformation. It is unnecessary to use the general expression (5) for this, it suffices to consider a particular case. For example, let $h_2=h_3=0$; h_1 is an arbitrary function. Then, from eqs.(5) and (6) one finds that $g_1 = h_1$, $g_2 = 1$, $g_3 = 0$. As both $A_i^a(h_\alpha)$ and $B_i^a(g_\alpha)$ are "spherical-symmetric" potentials, they can be related only by a spherical-symmetric gauge transformation with the matrix $U = e^{i\vec{\sigma}\vec{n}f(r)}$, here $\vec{\sigma}$ are the usual Pauli matrices. Let us assume that there exists such a function $f(r)$ by means of which one could make a transition from h_α to g_α, provided the following equalities are fulfilled:

$$\begin{cases} g_1 = h_1 \cos 2f + \frac{1}{2}\sin 2f = h_1 \\ g_2 = h_1 \sin 2f + \sin^2 f = 1 \\ g_3 = f' r = 0 \end{cases} \tag{6}$$

It is easy to see that the set (6) is unsolved and, thus, A_i^a and B_i^a cannot be reduced to each other under a gauge transformation. In the general case of arbitrary h_α, there are no gauge transformations relating $A_i^a(h_\alpha)$ and $B_i^a(g_\alpha)$. Of course, exceptions are possible. For example, the quadratic equation (4) can be degenerated, i.e. there exists the unique solution $g_2 = h_2$. And in this case there is also the gauge transformation relating two roots. This is the identical transformation with $U = 1$.

The condition of the existence of the unique solution of eq.(4) $g_2 = h_2$ can be written in the form:

$$(H_1^2 + H_2^2)(2h_2 - 1) + \frac{1}{2} H_1 r \frac{dH_3}{dr} = 0 . \tag{7}$$

So, for example, for $h_1 = h_3 = 0$, h_2 is an arbitrary func-

tion, the condition (7) is satisfied and, as readily see, we have the unique solution from eqs.(5) and (3): $g_2 = h_2$, $g_1 = g_3 = 0$.

3. It is interesting to know in what extent the potentials-doubles differ at $r \to \infty$. In the study of this question we confine ourselves to the systems with the finite total energy. To this end, it suffices that $h_\alpha \to const$ $(r \to \infty)$. Taking account of this assumption, one writes the solution of (5) in the form: $g_2 = 1 - h_2 - \Delta$, where $\Delta = \frac{1}{2}\frac{H_1}{H_1^2 + H_2^2} r \frac{dH_3}{dr} \to 0, (r \to \infty)$. For simplicity in calculations, the condition $h_1 = 0$ is imposed on the potential $A_i^a(h_\alpha)$. Then the solutions of eqs.(3) and (5) at $r \to \infty$ take the form:

$$g_1 \to 0, \quad g_2 \to 1 - h_2, \quad g_3 \to -h_3 . \tag{8}$$

But the potentials are not comparable in such a form since they are written in differential gauges. In order to compare $A_i^a(h_\alpha)$ and $B_i^a(g_\alpha)$, let us transit to the Coulomb gauge with the help of a spherical-symmetrical gauge transformation. $\tilde{h}_\alpha$ and $\tilde{g}_\alpha$ stand for the potentials h_α and g_α, respectively, which are reduced to the Coulomb gauge; here $U = e^{i\vec{\sigma}\vec{n}\varphi(r)}$ is the transformation matrix of h_α to $\tilde{h}_\alpha$ and, correspondingly, $S = e^{i\vec{\sigma}\vec{n}\psi(r)}$ is the transformation matrix of g_α to $\tilde{g}_\alpha$.

The conditions of transverseness of the potentials $A_i^a(\tilde{h}_\alpha)$ and $B_i^a(\tilde{g}_\alpha)$ which are written in the asymptotic region take the form:

$$\frac{d}{dr}(r^2\varphi') = -h_3 + \sin 2\varphi\,(1 - 2h_2) \tag{a}$$

(9)

$$\frac{d}{dr}(r^2\psi') = -g_3 + (1 - 2g_2)\sin 2\psi \tag{b}$$

Here at $\tilde{h}_\alpha$ $(z \to \infty)$ is expressed through φ and h_α as follows:

$$\tilde{h}_1 = \frac{1-2h_2}{2} \sin 2\varphi$$
$$\tilde{h}_2 = h_2 \cos 2\varphi + \sin^2\varphi \tag{10}$$
$$\tilde{h}_3 = h_3 + \varphi' z$$

and, similarly $\tilde{g}_\alpha$ through g_α and ψ :

$$\tilde{g}_1 = \frac{1-2g_2}{2} \sin 2\psi$$
$$\tilde{g}_2 = g_2 \cos 2\psi + \sin^2\psi \tag{11}$$
$$\tilde{g}_3 = g_3 + \psi' z$$

Remark that the equations of the type (9) have been considered by V.N.Gribov, though for another purpose. In ref. 8 the equations of the type (9) are interpreted as the equations describing the motion of a pendulum with friction for which the analogue of t is $\ln z/z_0$, the analogue of the force of gravity is $(1-2h_2)$, and that of h_3 is the force directed perpendicularly to the plane of motion. Taking into account eq.(8) and denoting the solution of eq.(9a) via φ_0 at $z \to \infty$, one can see that in the asymptotic region the solution of eq.(9b) tends to ψ_0 where

$$2\psi_0 = \pi - 2\varphi_0 . \tag{12}$$

Substituting eqs.(8) and (12) into eq.(11) and comparing with eq.(10), it is seen that at $z \to \infty$ the potentials-doubles in a Coulomb gauge are related as follows:

$$\tilde{g}_1 \to -\tilde{h}_1 , \quad \tilde{g}_3 \to -\tilde{h}_3 , \quad \tilde{g}_2 \to \tilde{h}_2 . \tag{13}$$

4. In this paragraph the following statement is proves: there are no potentials-doubles satisfying the Yang-Mills equations with the same sources in the class of spherical-symmetrical fields.

In the case of existence of such potentials, the following equality must take place:

$$\varepsilon^{abc} F_{ji}^{c} (A_j^b - B_j^b) = 0 , \tag{14}$$

which follows from the Yang-Mills equations. Using $\Delta_\alpha = g_\alpha - h_\alpha$, one rewrites eq.(14) in the form:

$$\begin{cases} \Delta_1 H_3 + \Delta_3 H_1 = 0 \\ \Delta_2 H_2 + \Delta_1 H_1 = 0 \\ \Delta_3 H_2 + \Delta_2 H_3 = 0 \end{cases} . \tag{15}$$

The system (15) can have the non-trivial solution if $H_3 (H_1^2 + H_2^2) = 0$. This condition is realized only in two cases:

a) $H_1 = H_2 = 0$.

The condition (7) is here fulfilled, whence, we have only the trivial solution: $g_\alpha = h_\alpha$.

b) $H_3 = 0$.

In this case eqs.(5) and (3) lead to the following expressions for g_α and Δ_α :

$$\begin{cases} g_2 = 1 - h_2 \\ g_1 = -h_1 \\ g_3 = -h_3 + \frac{h_2' z}{h_1} \end{cases} , \quad \begin{cases} \Delta_2 = 1 - 2h_2 \\ \Delta_1 = -2h_1 \\ \Delta_3 = -\frac{1}{h_1} H_1 \end{cases} \tag{16}$$

With the allowance for eq.(15) it follows that the solution with $g_\alpha = h_\alpha$ is possible only. Thus, the potentials-doubles describe a different distribution of the external sources.

Finally, note that in the four-dimensional case of a class of "axial-symmetric fields considered by V.N.Gribov in ref. 8 one can also obtain explicit expressions for the potentials-doubles.

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