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A b s t r a c t

It is shown that transformations admissible by the equations which are integrated by the method of inverse spectral transform form infinite-dimensional groups of different types, namely: the infinite-dimensional group of symmetry, the infinite-dimensional group of Backlund transformations, and the infinite-dimensional dynamical group.

ON A GROUP STRUCTURE OF TRANSFORMATIONS ADMISSIBLE BY INTEGRABLE EQUATIONS

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I. The Backlund transformations play an important role in a study of nonlinear equations. By now these transformations were derived for a wide class of equations. It was clarified the relationship between them and the method of inverse spectral transform. It was discussed the connection between the transformations of spectral data and the field quantities (see, e.g., refs. /1-6/). But the question on group properties of Backlund transformations is still open.

In the present note it is considered the group structure of transformations acting in a whole set of solutions both for an equation and a class of equations. It is shown that these transformations and, in particular, Backlund transformations form an infinite-dimensional group.

The solution space of nonlinear equations is a homogeneous space of the so-called dynamical group. This fact can be used for quantization of integrable equations by the method of quantization on orbits.

2. We shall consider the nonlinear equations which are integrable by the method of inverse spectral transform (IST). To this end, we shall use the version of this method proposed in the papers /5/. For the sake of simplicity, let us restrict ourselves to the class of equations which were studied in the first of those papers. The results obtained are valid in the general case as well. We shall make use of the notation of the paper /5/.

The relative linear problem is of the form:

$$\frac{\partial \psi}{\partial x} + ik\sigma_3 \psi = \frac{1}{2}\left[(v_1+v_2)\sigma_1 + i(v_1-v_2)\sigma_2\right]\psi \qquad (1)$$

where σ_1 , σ_2 , σ_3 are Pauli matrices, $v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is the two-component field quantity, k is the spectral parameter. Formula (1) and the equations of inverse transform determine a one-to-one correspondence between the functions $v(x)$ and the quantities ψ , and, hence, between the functions $v(x)$ and the spectral data $A(k)=\begin{pmatrix} \alpha^+(k) \\ \alpha^-(k) \end{pmatrix}$ and $B(k)=\begin{pmatrix} \beta^+(k) \\ \beta^-(k) \end{pmatrix}$*.

The equations which are integrable by the operator (1) may be written as follows /5/:

$$\sigma_3 \frac{\partial v(x,t,\vec{y})}{\partial t} + \vec{V}(L,t,\vec{y})\sigma_3 \frac{\partial v(x,t,\vec{y})}{\partial \vec{y}} + \gamma(L,t,\vec{y})v(x,t,\vec{y}) = 0 \quad (2)$$

where $\vec{V}$ and γ are the arbitrary functions and:

$$L = \frac{1}{2i}\sigma_3 \frac{\partial}{\partial x} - v I v^T \sigma_2 ,$$

$$(If)(x) = \int_x^\infty d\xi\, f(\xi,t,\vec{y}).$$

Eq. (2) is satisfied simultaneously with the following equation (plus the equation for $B(k,t,\vec{y})$):

$$\sigma_3 \frac{\partial A(k,t,\vec{y})}{\partial t} + \vec{V}(k,t,\vec{y})\sigma_3 \frac{\partial A(k,t,\vec{y})}{\partial \vec{y}} + \gamma(k,t,\vec{y})A(k,t,\vec{y}) = 0. \quad (3)$$

In virtue of the correspondence $v \leftrightarrow A$ each solution of the nonlinear equation (2) corresponds to the solution of linear equation (3) and vice versa.

Remark that if one proceeds from the spectral data $A(k,t,\vec{y})$ to Fourier transforms $\varphi(x,t,\vec{y})$ = $=\int dk\, exp(2ikx\sigma_3)\,A(k,t,\vec{y})$ then eq. (3) can be written in the form:

$$\sigma_3 \frac{\partial \varphi(x,t,\vec{y})}{\partial t} + \vec{V}\left(\frac{\sigma_3}{2i}\frac{\partial}{\partial x},t,\vec{y}\right)\frac{\partial \varphi(x,t,\vec{y})}{\partial \vec{y}} + \gamma\left(\frac{\sigma_3}{2i}\frac{\partial}{\partial x},t,\vec{y}\right)\varphi(x,t,\vec{y}) = 0 \quad (4)$$

* In this note we confine ourselves to the continuous spectrum. The case of discrete spectrum will be considered elsewhere.

Thus, the Fourier transforms of spectral data satisfy the equation which coincides with a linear part of the nonlinear equation (2).

3. Let us denote the set of solutions of eq. (2) as $\mathcal{M}_v$, and that of eq. (3) (or eq. (4)) as $\mathcal{M}_A$. The one-to-one correspondence between the elements of these sets permits us to determine the same correspondence between the operations with the elements. Indeed, each transformation $v \to v'$ in $\mathcal{M}_v$ by formulae (1) is mapped into transformation $A \to A'$ in $\mathcal{M}_A$, and inversely, each transformation in $\mathcal{M}_A$ by formulae of inverse spectral transform generates a transformation in $\mathcal{M}_v$.

There is no difficulty to see further that any one-parametric group of transformations in $\mathcal{M}_A$ corresponds to a one-parametric group of transformations in $\mathcal{M}_v$, and vice cersa. It is clear that this is also true for any finite-parametric group and any infinite-parametric one: each group of transformations in $\mathcal{M}_v$ induces a group of transformations in $\mathcal{M}_A$, which is isomorphic to it, and vice versa. Thus, the structures of the groups of transformations admissible by the sets $\mathcal{M}_v$ and $\mathcal{M}_A$ are the same. This circumstance makes it possible to reduce the question on the groups admissible by eq. (2) to the much more simpler case of linear equation (3), (4).

All the groups admissible by some equation can be divided into the groups of three types: a dynamical group, a group of symmetry, and a group of Backlund transformations. By the dynamical group (D) we mean a group of transformations which transforms any solution v of certain equation into any other solution v' of the same equation. The group of symmetry G , is a stationary group of solution $v = 0$*. The group of Backlund transformations (B) is a group of transformations inhomogeneous with respect to v . If an equation possesses the dynamical group D, then the set $\mathcal{M}_v$ of solutions of this equation is a homogeneous space of the group D, and $\mathcal{M}_v$ is isomorphic to the space of the left adjacent

* This definition is equivalent to the standard one.

classes of the group $\mathcal{D}$ under the subgroup G (group of symmetry).

Let us turn now to eqs. (3), (4). The listed above types of groups can be readily described for the case of a linear equation. The group, of symmetry is a group of transformations $\varphi \to \varphi' = \varphi + \delta\varphi$ (in the infinitesimal form) for which

$$\delta\varphi(x) = \varepsilon \mathcal{D}(x)\varphi(x)$$

and, in general, the integro-differential operator $\mathcal{D}$ commutes with the operator $\mathcal{F}$ ($[\mathcal{F}, \mathcal{D}] = 0$). In ref. (7) it was shown that the group of symmetry for an arbitrary linear equation is an infinite-dimensional Lie group and belongs to the $G_{n\infty}$ type. The existence of the infinite set of integrals of motion of a linear equation is associated with this fact.

The group of Backlund transformations of a linear equation is the group of transformations

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \omega(x)$$

where $\omega(x)$ is the arbitrary solution of this equation. It is easy to see that the group in question is an infinite-dimensional Abelian group /8/. The dynamical group is a group of transformations of the form $\delta\varphi(x) = \varepsilon\mathcal{D}(x)\,\omega(x)$ and, hence, this is a combination of the group of symmetry and the group of Backlund transformations. The dynamical groups were considered in considerable detail in ref. /8/. It has turned out that eq. (4) can be derived from the dynamical group D by the method of nonlinear realizations /9/.

As we see, the groups admissible by eq. (2) are isomorphic to those admissible by eqs. (3), (4). Thereby, eq. (2) possesses the infinite-dimensional Abelian group of Backlund transformations. For arbitrary functions $\vec{V}$ and γ eq.(2) possesses the infinite-dimensional group of symmetry $G_{1\infty}$ connected to an invariance of this equation under the translations $x \to x' = x + a$. The definite infinite-dimensional groups of symmetry of the $G_{n\infty}$ type * correspond to the

* Note that the spectral parameter k possesses definite transformative properties under the group of symmetry: it is transformed as $\sigma_3 \frac{\partial}{\partial x}$ /7/.

definite functions $\vec{V}$ and γ. Thorough a whole set* $\mathcal{M}_v$ of solutions of eq. (2) the infinite-dimensional dynamical group acts and the set $\mathcal{M}_v$ is a homogeneous space of this group. Since using the transformations of a dynamical group, it is possible to obtain all the solutions from any solution of the equation (in particular, from initial data), the information which is contained in the dynamical group coincides completely with that in the equation.

The fact that the solution space of eq. (2) is a homogeneous space of dynamical group D (i.e. the orbit of group D) gives a possibility, as it seems to us, to quantize eq. (2), using the method of quantization on orbits (see, e.g., ref. /10/).

And note finally, that the group properties which are intrinsic to the solutions of eq. (2), corresponding to the continuous spectrum of operator (1) coincide, in essence, with the group properties of the linear part of equation. This circumstance may help in finding the "group" definition of the solutions of continuous spectrum. The solutions of eq. (2) which are described by the continuous spectrum (non-soliton solutions) are the solutions whose group properties (group of symmetry, group of Backlund transformations, and dynamical group) coincide with the group properties of solutions of the linear part of equation.

4. The Backlund transformations and the transformations of the group of symmetry which have a simple form in the space of spectral data are quite complex in the field variable $v(x,t,\vec{y})$ /7,8/. For the (2)-type equations the transformations, in a particular case $\vec{V} = 0$, were derived, in essence, in ref. /5/. In this paper it was shown that the transformation $v \to v'$ of the form

$$\left(g(\Lambda) + f(\Lambda)\sigma_3\right)v' + \left(g(\Lambda) - f(\Lambda)\sigma_3\right)v = 0 \quad (5)$$

where $g(z)$, $f(z)$ are the arbitrary functions, and $\Lambda = \Lambda(v', v)$ is some integro-differential operator, transforms the solution

* Recall that the functions $\vec{V}$ and γ are arbitrary and we confine ourselves only to the continuous spectrum.

v of eq. (2) ($\vec{V}$ = 0) into the solution v' of the same equation. The quantity $A(k,t)$ is transformed as follows:

$$A(k,t) \to A'(k,t) = \begin{pmatrix} \frac{f(k)-g(k)}{f(k)+g(k)}, & 0 \\ 0, & \frac{f(k)+g(k)}{f(k)-g(k)} \end{pmatrix} A(k,t) \quad (6)$$

Let us introduce the function $\rho(z) = 2\,\mathrm{arcth}\left(-\frac{g(z)}{f(z)}\right)$

Then transformations (5) and (6) take the form:

$$v'(x,t) = \exp\left(\rho(\Lambda(v',v))\sigma_3\right) \cdot v(x,t), \quad (7)$$

$$A'(k,t) = \exp\left(\rho(k)\sigma_3\right) A(k,t).$$

It is ready to see from expressions (7) that transformations under consideration form an infinite-dimentional Lie group: the function ρ is a transformation parameter; ρ = 0 corresponds to the identical transformation, the function $\rho = 0$ corresponds to the identical transformation, the function $-\rho$ corresponds to the inverse transformation.

The case $\vec{V} = 0$ is degenerate. In the general case $\vec{V} \neq 0$ transformations (7), form merely a small subgroup of the whole dynamical group.

In ref. /5/ transformations (5) and (6) with the function dependent on t were also considered. These transformations, the so-called generalized Backlund transformations, transform the solutions of an equation of type (2) ($\vec{V}$ = 0) into the solution of another equation of type (2) (with another function γ). It is evident that the generalized Backlund transformations form a group. In the space which is a combination of the spaces of the solutions of eq. (2) with all possible functions γ , the generalized dynamical group can be determined so that the united space of solutions is a homogeneous space of this group. The generalized Backlund transformations give a possibility to obtain the

functional relations such that the initial differential equation is their infinitesimal form /5/.

The results presented above are also true for any equation integrable by the IST method.

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