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И Н С Т И Т У Т
ЯДЕРНОЙ ФИЗИКИ СОАН СССР

ПРЕПРИНТ И Я Ф 77 - 7

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Новосибирск
1977

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A b s t r a c t

In the frame of operator technique the electron operator Green's function has been found. Conclusion is based on a fact that some angular operators for the Dirac equation form the quaternion group. Matrix element of the operator Green's function found coinside with standard electron Green's function which was calculated previously.

The operator diagram technique within the framework of which a number of important results have been recently obtained seems to be very useful for consideration of radiative effects in external fields. By the present time the phenomena in a uniform, constant electromagnetic field, the field of a plane wave and their superpositions /1-3/ have been studied. It is believed that this method will be also very valuable for investigation of the corresponding problems in the Coulomb field (the Lamb-shift in heavy atoms, the vacuum polarization in the strong Coulomb field).

In this work electron operator Green's function in the Coulomb field $\hat{G}$ has been found which is one of the components of the approach mentioned above. The value $\langle \vec{r}' | \hat{G} | \vec{r} \rangle$ agrees with Green's function discussed in a variety of works (see, for instance, /4,5/ and cited in these works).

Electron operator Green's function for an central-symmetrical field is written as follows

$$\hat{G} = \frac{1}{\gamma^0(\varepsilon - \mathcal{U}(r)) - \vec{\gamma}\vec{p} - m + i0} \tag{1}$$

where $\mathcal{U}(r)$ is a potential; $\mathcal{U}(r) = -\frac{Z\alpha}{r}$ in the attractive Coulomb field. The operator $\vec{\gamma}\vec{p}$ in the expression (1) can be represented in the known form

$$\vec{\gamma}\vec{p} = \left[p_r - \frac{i}{r}(1 + \vec{\Sigma}\vec{L})\right]\vec{\gamma}\vec{n} \tag{2}$$

where $\vec{L}$ is the operator of an angular momentum, $\vec{n} = \vec{z}/|\vec{z}|$, $p_z = -\frac{i}{z}\frac{\partial}{\partial z}z$. Using this formula, taking out the combination $\gamma\vec{n}$ out of the denominator $\hat{G}$, and carrying out the exponential parametrization, we obtain

$$\hat{G} = -i\int_C ds\, \gamma\vec{n} \exp\Big\{is\Big[p_z - \frac{i}{z}(1+\vec{\Sigma}\vec{L}) + [\gamma^0(\varepsilon - U(z)) - m]\gamma\vec{n}\Big]\Big\} \tag{3}$$

where the integration contour C is going from zero to infinity in a complex plane in such a way that integral (3) would exist; it should be borne in mind that the exponent in the intergrand (3) contains γ -matrices.

Let us consider now the combination

$$f = \exp\Big\{is\Big[p_z - \frac{i}{z}(1+\vec{\Sigma}\vec{L}) + [\gamma^0(\varepsilon - U(z)) - m]\gamma\vec{n}\Big]\Big\} e^{-isp_z} \tag{4}$$

Differentiating this expression with respect to s and taking into account that

$$e^{isp_z}\varphi(z)e^{-isp_z} = \varphi(z+s) \tag{5}$$

is valid for an arbitrary function $\varphi(z)$, we find

$$if^{-1}\frac{df}{ds} = \frac{i}{z+s}(1+\vec{\Sigma}\vec{L}) - [\gamma^0(\varepsilon - U(z+s)) - m]\gamma\vec{n} \tag{6}$$

From this equation the operator function f is seen to contain only angular operators and not to depend on p_z. The solution of the equation (6) can be represented in the form

$$f = T_{(-)} \exp\Big\{-i\int_0^s ds\Big[\frac{i}{z+s}(1+\vec{\Sigma}\vec{L}) - [\gamma^0(\varepsilon - U(z+s)) - m]\gamma\vec{n}\Big]\Big\} \tag{7}$$

here the symbol $T_{(-)}$ denotes the antichronologic operator product in respect to "time" s.

It turns out that the $T_{(-)}$ product in formula (7) can be found in the explicit form. This is due to the fact that the operators

$$K_0 = \gamma\vec{n}, \quad K_1 = \frac{1+\vec{\Sigma}\vec{L}}{\sqrt{\vec{J}^2+\frac{1}{4}}}, \quad K_2 = K_0K_1 \tag{8}$$

where $\vec{J}^2 = \left(\vec{L} + \frac{\vec{\Sigma}}{2}\right)^2$ commutes with all K_n, form the group

$$K_0^2 = -1, \quad K_1^2 = K_2^2 = 1;$$

$$K_0K_1 = -K_1K_0 = K_2, \tag{9}$$

$$K_2K_0 = -K_0K_2 = K_1, \quad K_1K_2 = -K_2K_1 = -K_0$$

what it is easily to see taking into account that an anticommutator $\{\gamma\vec{n}, (1+\vec{\Sigma}\vec{L})\} = 0$. Note that the operators K_0, iK_1, iK_2 (adding the unit element) form the algebra of quaternions.

Making use of relations (9) we shall represent the solutions of equation (6) in the following form

$$f = f_0 K_0 + f_1 K_1 + f_2 K_2 + f_3 \tag{10}$$

After substituting the expression (10) into equation (6) we get the system of equations

$$\begin{aligned}
\frac{df_0}{dt} &= \frac{\eta}{t} f_2 + D_- f_3 \\
\frac{df_1}{dt} &= \frac{\eta}{t} f_3 - D_+ f_2 \\
\frac{df_2}{dt} &= \frac{\eta}{t} f_0 - D_- f_1 \\
\frac{df_3}{dt} &= \frac{\eta}{t} f_1 + D_+ f_0
\end{aligned} \tag{11}$$

where $D_\pm = i[\gamma^0(\varepsilon - U(t)) \pm m]$, $t = z + s$, $\eta = \sqrt{\vec{J}^2 + \frac{1}{4}}$. This system must be solved under the following boundary conditions

$$f(s=0) = 1, \quad f'(s=0) = \frac{\eta}{z} K_1 + D K_0 \tag{12}$$

One can easily verify that for the combinations $f_0 \pm f_2$ and $-i(f_3 \mp f_1)$ the system (11) reduces to the system of Dirac equations for radial functions in a central-symmetric field $U(z)$. In the following let us consider the attractive Coulomb field. Using the known form of wave functions (see, e.g., /6/), expressing them through the Whittaker functions M and W (determined as in /7/), and satisfying the boundary conditions (12), we get the explicit form of the function f (10).

From the formula (3), (4), (8) we have

$$\hat{G} = -i \int_C K_0 f e^{isp_z} ds$$

Substituting here the obtained expression for f, taking (9) into account and performing not complicated transformations with making of use the recurrent relations for the Whittaker functions, we obtain the following final expression for the electron Green's function in the Coulomb field:

$$\hat{G} = -\frac{\Gamma(\gamma-\nu)}{2\Gamma(2\gamma+1)} \Big\{ N_+(\rho_0) \int_0^\infty \frac{1}{\sqrt{\rho\rho_0}} \Big[\frac{m+\varepsilon\gamma^0}{\lambda} V_+(\rho) + i V_-(\rho) K_0 \Big] e^{isp_z} + V_+(\rho_0) \int_{-\infty}^0 \frac{1}{\sqrt{\rho\rho_0}} \Big[\frac{m+\varepsilon\gamma^0}{\lambda} N_+(\rho) - i N_-(\rho) K_0 \Big] e^{isp_z} ds \Big\} \tag{13}$$

where

$$N_\pm(\rho) \equiv (\gamma-\nu) M_{\nu-\frac{1}{2},\gamma}(\rho) \pm \Big(\eta K_1 + \frac{Z\alpha m\gamma^0}{\lambda}\Big) M_{\nu+\frac{1}{2},\gamma}(\rho)$$

$$V_\pm(\rho) \equiv W_{\nu+\frac{1}{2},\gamma}(\rho) \pm \Big(-\eta K_1 + \frac{Z\alpha m\gamma^0}{\lambda}\Big) W_{\nu-\frac{1}{2},\gamma}(\rho) \tag{14}$$

here $\eta = \sqrt{\vec{J}^2 + \frac{1}{4}}$, $\gamma = \sqrt{\eta^2 - Z^2\alpha^2}$, $\lambda = \sqrt{m^2 - \varepsilon^2}$, $\nu = \frac{Z\alpha\varepsilon}{\lambda}$, $\rho^0 = 2\lambda z$, $\rho = 2\lambda(z+s)$; $M_{\nu\pm\frac{1}{2},\gamma}(\rho)$, $W_{\nu\pm\frac{1}{2},\gamma}(\rho)$ are the Whittaker functions.

The choice of the integration contour C is carried out with taking the known asymptotic properties of the Whittaker functions into consideration.

To find the standard form of the Green's function one should keep in mind that $\langle \vec{z} | \hat{G} | \vec{z}' \rangle \equiv \hat{G}\, \delta(\vec{z}-\vec{z}')$,

$$e^{isp_z} = \left(1 + \frac{s}{z}\right) e^{s\partial_z} \tag{15}$$

and also that operator $e^{s\partial_z}$ is the displacement operator. Representing

$$\delta(\vec{z}-\vec{z}') = \frac{\delta(z-z')}{zz'}\,\delta(\vec{n}-\vec{n}')$$

and taking (15) into account, we have

$$e^{isp_z}\,\frac{\delta(z-z')}{zz'} = \frac{\delta(z+s-z')}{zz'} \tag{16}$$

Using the known expansion of an angular δ-function in spherical harmonics, substituting (16) into (13) and carrying out the integration over s, we get the common representa - tion of the Green's function (see, for example, /4/).

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Работа поступила - 8 декабря 1976 г.

Ответственный за выпуск - С.Г.ПОПОВ
Подписано к печати 25.I-1977 г. МН 02621
Усл. 0,3 печ.л., 0,2 учетно-изд.л.
Тираж 200 экз. Бесплатно
Заказ № 7.

Отпечатано на ротапринте ИЯФ СО АН СССР