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ON THE THEORY OF MAGNETIC CHARGE

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In the precedinge author's paper/1/ the nonrelativistic theory of interaction of electric and magnetic charges has been proposed in which the notion is not used about a monopole magnetic field singular on a string. Such a field has been introduced in the classical Dirac work/2/ to represent the interaction of electric charge and monopole in usual minimal form

$$\mathcal{L}_{int} = e\bar{v}\bar{A} \qquad (1)$$

($\bar{v}$ - electric charge velocity, $\bar{A}$ - vector potential of monopole field. For simplicity we shall concider an infinitely heavy monopole locating in the origin of coordinates).

It is clear, that Coulomb magnetic field can't be represented in whole spase as a rotor of some vector $\bar{A}$. Nevertheless, as Dirac has showed, one can get it everywhere but some line (string) , where the vector potential is singular and magnetic field differs from true monopole field. The usage of magnetic field containing nonphysical and singular on string contribution is an essential feature of standard Dirac monopole theory. This feature has leaded to a number of theoretical difficulties which are rather often discussed in literature.

In paper/1/ the Lagrange function of electric charge in monopole field has been introduced leading to the correct magnetic Lorentz force in all points of space without any exception. This function has a form

$$\mathcal{L} = \frac{\mu \dot{r}^2}{2} + \frac{\mu r^2}{2}\left[\bar{n}\bar{\Omega}\right]^2 - \frac{eg}{4\pi}\bar{n}\bar{\Omega} \qquad (2)$$

so that Lagrangian of interaction

$$\mathcal{L}_{int} = -\frac{eg}{4\pi}\bar{n}\bar{\Omega} \qquad (3)$$

is used instead of (1). Here $\bar{r} = \bar{n}r$ - radius-vector of electric charge, $\bar{\Omega}$ - angular velocity of a reference frame in which the charge and monopole are motionless. The orientation of axes of this frame with respect to axes of the laboratory one is given in each moment by Euler angles α, β, γ, which together with distance r are the generalized coordinates of problem. If z - axis of a moving frame is chosen along vector $\bar{n}$, one obtains

$$\mathcal{L} = \frac{\mu\dot{r}^2}{2} + \frac{\mu r^2}{2}\left(\dot{\beta}^2 + \sin^2\beta\,\dot{\alpha}^2\right) - \frac{eg}{4\pi}\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right) \qquad (4)$$

The angles α and β coincide with the spherical coordinates θ and φ. At the same time the variation over angle γ provides a trivial relationship $\frac{d}{dt}\left(\frac{eg}{4\pi}\right) = 0$ only. That is why the term with $\dot{\gamma}$ can be omitted in classical theory. The meaning of this term is clarified during quantization/1/. The canonical impulse $P_\gamma = \frac{\partial\mathcal{L}}{\partial\dot{\gamma}}$ has a meaning of projection $\bar{n}\bar{J}$ of angular momentum $\bar{J}$ on the direction of radius-vector $\bar{r}$. The difference of this projection from zero is a specific feature of the motion in monopole field and its quantization leads to the known Dirac condition

$$-\frac{eg}{4\pi} = \frac{1}{2}n \quad (n = 0, \pm 1, \pm 2, \dots) \qquad (5)$$

Thus (4) is in essence the Lagrange function in spherical coordinates. Its variation over r, β and α provides the true

classical equations of motion for a charge in Coulomb monopole magnetic field expressed though these coordinates.

The usage of spherical coordinates however is inconvenient for transition to a relativistic theory, when in particular the radiation field should be taken into account, as well as for generalization of the theory for a system of several charges and monopoles. In this note the formulation of the theory and its quantization in Cartesian coordinates are cosidered. The connection between interection Lagrangians (1) and (3) is analyzed in more detail than in/1/. As it will be seen, in classical theory these Lagrangians practically coincide, if the vector potential $\bar{A}$ in (1) is chosen in Schwinger/3/ form

$$\bar{A} = \frac{g}{4\pi}\bar{a}$$

$$\bar{a} = \frac{1}{2r}\left(\frac{[\bar{r}\bar{\nu}]}{r - \bar{r}\bar{\nu}} - \frac{[\bar{r}\bar{\nu}]}{r + \bar{r}\bar{\nu}}\right) = (\bar{n}\bar{\nu})\frac{1}{r}\frac{[\bar{n}\bar{\nu}]}{[\bar{n}\bar{\nu}]^2} \qquad (6)$$

with line of singularity along z axis of lab. frame. ($\bar{\nu}$ - a unit vector in the direction of z axis).

On the other hand, Lagrange equations change when transforming from spherical to Cartesian coordinates due to a multiple-valued dependance of angle α on a point of space. Lagrange equations in Cartesian coordinates are nonhomogeneous and contain in the right side the additional singular on z axis "force". This "force" has completely compensated the singular part of mag. force corresponding to potential (6). As a resalt the equations of motion in Cartesian coordinates contain no terms singular on z axis.

The existence of an additional "force" also influences the form of Hamiltonian equation and Poisson brackets of Cartesian components of canonical impulse. However the equations of motion have an ordinary form when written by means of Poisson brackets. This circumstance allows quantization directly in Cartesian coordinates. The essential peculiarity of quantization of charge motion in monopole field is noncommutativity of components of operator $\hat{P} = -i\bar{\nabla}$ on the system's wave function.

2. Let us make transition from Lagrange equations in curvilinear coordinates $\xi_i = (r, \beta, \alpha)$ to equations in Cartesian ones $x_i = (x, y, z)$. For this purpose let us firstly express the Lagrang function (4) in Cartesian coordinates. The term with $\dot{\gamma}$ unimportant in a classical theory is further omitted. Due to $\dot{\xi}_i = \bar{v}\bar{\nabla}\xi_i$ we find

$$\dot{r} = \bar{v}\bar{n} \quad ; \quad \dot{\beta} = \bar{v}\frac{[\bar{n}[\bar{n}\bar{\nu}]]}{r|[\bar{n}\bar{\nu}]|} \quad ; \quad \dot{\alpha} = -\bar{v}\frac{[\bar{n}\bar{\nu}]}{r[\bar{n}\bar{\nu}]^2} \tag{7}$$

Substituting this in (4) we obtain

$$\mathcal{L}[\xi(x), \dot{\xi}(x,\dot{x})] \equiv \tilde{\mathcal{L}}(x,\dot{x}) = \frac{\mu\bar{v}^2}{2} + e\bar{v}\bar{A} \tag{8}$$

where

$$\bar{A} = -\frac{g}{4\pi}(\bar{n}\bar{\nu})\bar{\nabla}\alpha = \frac{g}{4\pi}\bar{a} \tag{9}$$

Consequently the interaction Lagrangian (3) has a minimal form in Cartesian coordinates if the vector $\bar{A}$ is being interpreted as vector potential of monopole magnetic field. This potential coincides with Schwinger potential (6) and is singular on z axis. It is clear however that this singularity is due to the purely kinematic singularity of angular velocity $\dot{\alpha}$ and has no relation to real monopole magnetic field. Therefore it must disappear from equation of motion. We shall prove this with help of direct transformation of coordinates in Lagrange equations.

As

$$\frac{\partial\tilde{\mathcal{L}}}{\partial x_i} = \frac{\partial\mathcal{L}}{\partial\xi_j}\frac{\partial\xi_j}{\partial x_i} + \frac{\partial\mathcal{L}}{\partial\dot{\xi}_j}\frac{\partial\dot{\xi}_j}{\partial x_i} = \frac{\partial\mathcal{L}}{\partial\xi_j}\frac{\partial\xi_j}{\partial x_i} + \frac{\partial\mathcal{L}}{\partial\dot{\xi}_j}\frac{\partial^2\xi_j}{\partial x_i\partial x_k}\dot{x}_k$$

and

$$\frac{d}{dt}\frac{\partial\tilde{\mathcal{L}}}{\partial\dot{x}_i} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_j}\frac{\partial\dot{\xi}_j}{\partial\dot{x}_i}\right) = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\xi}_j}\right)\frac{\partial\xi_j}{\partial x_i} + \frac{\partial\mathcal{L}}{\partial\dot{\xi}_j}\frac{\partial^2\xi_j}{\partial x_k\partial x_i}\dot{x}_k$$

then

$$\frac{d}{dt}\frac{\partial\tilde{\mathcal{L}}}{\partial\bar{v}} - \frac{\partial\tilde{\mathcal{L}}}{\partial\bar{r}} = -\left[\bar{v}\,\mathrm{rot}(\bar{\nabla}\xi_i)\right]\frac{\partial\mathcal{L}}{\partial\dot{\xi}_i} \equiv \bar{Q} \tag{10}$$

In the time of derivation of (10) we have taken into account that in curvilinear coordinates

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\xi}_i} - \frac{\partial\mathcal{L}}{\partial\xi_i} = 0 \tag{11}$$

It is important to emphasize once more that equations (11) with Lagrang function (4) do not contain a singular magnetic force.

As follows from (I0), Lagrange equations in Cartesian coordinates are homogeneous, if $\bar{Q} \equiv 0$ only. However it is clear that $\bar{Q} \neq 0$ in our case. Indeed $\operatorname{rot} \bar{\nabla} \alpha \neq 0$ as the angle α is no a single-valued point function. It may be verified by means of calculation of $\operatorname{rot} \bar{\nabla} \alpha$ flow through arbitrarily small surface crossed by z axis. By Stocks theorem this flow coincides with circulation of vector $\bar{\nabla} \alpha$ around z axis. This circulation is equal to 2π, so that

$$\operatorname{rot} \bar{\nabla} \alpha = 2\pi \bar{\nu} \delta(\bar{r}_\perp) \tag{I2}$$

where $\bar{r}_\perp$ - a component of radius-vector orthogonal to z axis. As a resalt the additional "force" $\bar{Q}$ is found to be equal

$$\bar{Q} = \frac{eg}{4\pi} (\bar{n}\bar{\nu}) [\bar{v} \operatorname{rot} \bar{\nabla} \alpha] \tag{I3}$$

(In this relation it has been taken into account, that $\bar{r}_\perp \delta(\bar{r}_\perp) = 0$ and that velocity $\bar{v}$ has no singularity on z axis.)

On the other hand due to (9)

$$\operatorname{rot} \bar{A} = \frac{g}{4\pi} [\bar{\nabla} \alpha \times \operatorname{grad}(\bar{n}\bar{\nu})] - \frac{g}{4\pi} (\bar{n}\bar{\nu}) \operatorname{rot} \bar{\nabla} \alpha =$$

$$= \frac{g}{4\pi} \frac{\bar{n}}{r^2} - \frac{g}{4\pi} (\bar{n}\bar{\nu}) \operatorname{rot} \bar{\nabla} \alpha \tag{I4}$$

As a consequence of (I3) and (I4) the terms singular on z axis cancel and the equations of motion take the form

$$\mu \frac{d\bar{v}}{dt} = \frac{eg}{4\pi} \frac{1}{r^2} [\bar{v} \bar{n}] \tag{I5}$$

as it should be in the case of motion of electric charge in magnetic field $\bar{\mathcal{H}} = \frac{g}{4\pi} \frac{\bar{n}}{r^2}$. Let us note that if $\bar{Q}$ is omitted in the right side of (I0) we shall obtain the equations of motion in a field with a string along z axis.

3. The Hamiltonian function in Cartesian coordinates has the form

$$H[\xi(x), \pi(x,p)] \equiv \tilde{H}(x,p) = \frac{(\bar{p} - e\bar{A})^2}{2\mu} \tag{I6}$$

where

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\xi}_i}, \quad p_i = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}_i} = \frac{\partial \xi_j}{\partial x_i} \pi_j \tag{I7}$$

(π_i - a canonical impulse in spherical coordinates).

With the help of (I0) and Lagrange transformation we can obtain

$$\frac{dx_i}{dt} = \frac{\partial \tilde{H}}{\partial p_i}$$

$$\frac{dp_i}{dt} = -\frac{\partial \tilde{H}}{\partial x_i} + Q_i \tag{I8}$$

so that Hamiltonian equations do not have a canonical form. In consequence the contribution of string drops out from equations

(I8) and they become completely equivalent to (I5).

Now let us find the Poisson brackets of X and p . We can compute them knowing, how these quantities depend on canonical variables ξ and π . After simple calculations we obtain

$$\{X_i, X_j\} = 0 \ , \quad \{P_i, X_j\} = \delta_{ij} \tag{I9}$$

but

$$\{P_i, P_j\} = \left(\frac{\partial^2 \xi_k}{\partial X_i \partial X_j} - \frac{\partial^2 \xi_k}{\partial X_j \partial X_i}\right)\pi_k = -\frac{eg}{4\pi}(\bar{n}\bar{\nu})\varepsilon_{ijk}\, \mathrm{rot}_k \bar{\nabla}\alpha \tag{20}$$

so strictly speaking Cartesian impulses are not canonical. Using (I9) and (20) equations (I8) may be represented in the form

$$\frac{dX_i}{dt} = \{\tilde{H}, X_i\}$$
$$\frac{dP_i}{dt} = \{\tilde{H}, P_i\} \tag{2I}$$

Finally because of (I9), (20)

$$\{v_i, v_j\} = \frac{1}{\mu^2}\{P_i - eA_i, P_i - eA_j\} = -\frac{eg}{4\pi}\frac{1}{\mu^2}\varepsilon_{ijk}\left[(\bar{n}\bar{\nu})\,\mathrm{rot}_k \bar{\nabla}\alpha - \mathrm{rot}_k\left((\bar{n}\bar{\nu})\bar{\nabla}\alpha\right)\right] = -\frac{eg}{4\pi}\frac{1}{\mu^2}\varepsilon_{ijk}\frac{n_k}{r^2} = -\frac{e}{\mu^2}\varepsilon_{ijk}\mathcal{H}_k \tag{22}$$

The contribution of string cancels in (2I) and (22).

Similarly the singular parts disappear during the calculation of Poisson brackets of others mechanical quantities, for example of components of angular momentum /I/

$$\bar{J} = [\bar{r}(\bar{p} - e\bar{A})] - \frac{eg}{4\pi}\bar{n}$$

Formulas (I9) - (2I) provide a foundation for quantization.

4. The quantization is carried out by means of substitutions:

$$\pi_i \to \hat{\pi}_i = -i\frac{\partial}{\partial \xi_i} \ , \quad -\frac{eg}{4\pi} \to \hat{P}_\gamma = -i\frac{\partial}{\partial \gamma}$$
$$P_i \to \hat{P}_i = \frac{\partial \xi_j}{\partial X_i}\hat{\pi}_j = -i\frac{\partial}{\partial X_i} \tag{23}$$

The Hamiltonian operator is equal to

$$\hat{\tilde{H}} = \frac{(\hat{\bar{p}} + \bar{a}\hat{P}_\gamma)^2}{2\mu} \tag{24}$$

and commutes with impuls $\hat{P}_\gamma$.

As noted above, the operator $\hat{P}_\gamma$ has a meaning of an operator of the angular momentum projection $\bar{n}\hat{\bar{J}}$ on the direction of radius-vector. In classical case this projection has a single value $-\frac{eg}{4\pi}$ for definit system. That is why it is natural to suppose, that in quantum mechanics the superselection rule exists for quantum number P_γ and only states with fixed value of P_γ are realized. The possible values of P_γ are given by formula (5). Therefore we can make use of Hamiltonian

$$\hat{\tilde{H}} = \frac{(\hat{\bar{p}} - \frac{1}{2}n\bar{a})^2}{2\mu} \tag{25}$$

The quantum mechanical equation of motion are obtained from (2I) by means of substitution of classical Poisson brackets by quantum ones. The commutators of operators X and $\hat{P}$ may be found with the help of the formulas (23). It is easy to verify, that as usual

$$[x_i, x_j] = 0 \quad , \quad [x_i, p_j] = i\delta_{ij} \qquad (26)$$

On the other hand

$$[\hat{p}_i, \hat{p}_j] = -i\varepsilon_{ijk}\, \mathrm{rot}_k \bar\nabla\alpha\, \hat{\pi}_\alpha \qquad (27)$$

The wave function of charge in monopole magnetic field may be represented in general case in form of superposition of stationary states, their angular part being given[2] by Wigner function $\mathcal{D}^j_{m'm}(\alpha,\beta,\gamma) = e^{im'\gamma} d^j_{m'm}(\beta) e^{im\alpha}$. In virtue of superselection rule for P_γ all terms in this superposition correspond to one and the same number m', so dependence of wave function on angle γ is always given by phase factor $e^{im'\gamma}$. Using the known[4] property of d - functions

$$d^j_{m'm}(0) = \delta_{m'm} \quad , \quad d^j_{m'm}(\pi) = (-1)^{j+m}\delta_{m'-m} \qquad (28)$$

we can easily conclude that*)

$$\mathrm{rot}(\bar\nabla\alpha)\, \hat{\pi}_\alpha \mathcal{D}^j_{m'm} =$$

$$= (\bar{n}\bar\nabla)\, \mathrm{rot}(\bar\nabla\alpha)\, \hat{P}_\gamma \mathcal{D}^j_{m'm} = -\frac{eg}{4\pi}(\bar{n}\bar\nabla)\, \mathrm{rot}\, \bar\nabla\alpha\, \mathcal{D}^j_{m'm} \qquad (29)$$

$$-\frac{eg}{4\pi} = m' = \frac{1}{2}n$$

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*) To compare with ordinary situation note, that for example in the case of motion in electric central field an angular dependence of wave function is written by spherical harmonics $Y_{em}(\beta,\alpha)$, vanishing on z axis at $m \neq 0$, so that

$$\mathrm{rot}(\bar\nabla\alpha)\, \hat{\pi}_\alpha Y_{em}(\beta,\alpha) \equiv 0$$

Consequently for arbitrary state of system under consideration

$$[\hat{p}_i, \hat{p}_j] = i\frac{eg}{4\pi}(\bar{n}\bar\nabla)\varepsilon_{ijk}\, \mathrm{rot}_k \bar\nabla\alpha \qquad (30)$$

$$-\frac{eg}{4\pi} = \frac{1}{2}n$$

This relation is a quantum version of (20). That is why the nonphysical contribution of string will cancel in the time of calculation of commutators of any physical quantities in quantum case too.

Using (30) we can obtain as one would think

$$\varepsilon_{ijk}[\hat{p}_i[\hat{p}_j, \hat{p}_k]] = 2eg\,\delta(\bar{r}) \qquad (3I)$$

On the other hand, the left side of (3I) is equal to zero identically (Jacoby identity). However as the radial dependence of stationary states of charge in monopole field is given[5] by Bessel functions

$$R_j(r) = \frac{\mathrm{const}}{\sqrt{r}} J_{\sqrt{(j+\frac{1}{2})^2 - (\frac{n}{2})^2}}(kr) \qquad (32)$$

($\frac{1}{2}n \leq j$ - half-integer), vanishing in the origin of coordinates at any $n \neq 0$ and j , Jacoby identity is fulfiled for any wave function of the system.

The above consideration shows, that Dirac string has a purely kinematic origin. Its cotribution disappeared from both equation of motion and Poisson brackets (classical and quantum) of any physical quantity. Therefore it can't lead to any absorvable consequences. Our approach makes superfluous an introduction of a special condition of "Dirac veto" type. Similarly in contrast with some times expressed opinion the Dirac quantization rule (5) is not connected with existence of string (but is due to quantization of internal projection of angular momentum).

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