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A b s t r a c t

It is shown that there no exist finite continuous groups of transformations in the Minkowsky space-time which contain the conformal group as a subgroup.

On the nonexistence of a symmetry
higher than conformal

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The Lorentz and Poincare groups play as known an important role as the groups of a symmetry of the quantum field theory. Lately there exists considerable interest also to the conformal group, which is an exact group of symmetry for massless particles and a broken group of symmetry for massive particles. The remarkable feature of the conformal symmetry is that it fixes the form of two- and threepoint functions independently of the concrete dynamics [1,2]. It would be attractive to find such group of transformations in the Minkowsky space-time which would define a form of the fourpoint functions and so on. Independently of the question of the fixing of a part of the dynamics by a group of the symmetry it is interesting to study possible space-time symmetries higher than Poincare and conformal.

The purpose of this paper is to show that there no exist finite continuous groups (Lie groups) of transformations in the Minkowsky space-time containing a subgroup which is locally isomorphic to the conformal group. Therefore, there no exists finite continuous group of space-time symmetry higher than conformal.

The generators of the continuous group of transformations in Minkowsky space-time have as known the form

$$L_i(x) = i f_{i\mu}(x) \frac{\partial}{\partial x^\mu}, \quad \mu = 0,1,2,3; \; i = 1,\dots, N; \tag{1}$$

where the functions $f_{i\mu}(x)$ are determined from the transformation law of coordinates. In particular, for generators of the conformal group we have [3]:

$$M_{\mu\nu} = i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\right), \quad P_\mu = i\frac{\partial}{\partial x^\mu},$$

$$K_\mu = i(x^2 g_{\mu\nu} - 2x_\mu x_\nu)\frac{\partial}{\partial x^\nu}, \quad \mathcal{D} = i x_\rho \frac{\partial}{\partial x^\rho} \tag{2}$$

We will show that in the Minkowsky space-time it isn't possible to realize the algebra of the finite continuous group which has the generators of form (1) and contain the conformal group as a subgroup.

It is convenient to use the isomorphism between the conformal group and the group $SO(2,4)$. The generators L_{ab} (a,b = 0,1,2,3,5,6) of the group $SO(2,4)$ and generators of the conformal group are connected by the relations [3]:

$$L_{\mu\nu} = M_{\mu\nu}, \quad L_{5\mu} = \frac{1}{2}(P_\mu + K_\mu),$$

$$L_{6\mu} = \frac{1}{2}(P_\mu - K_\mu), \quad L_{56} = \mathcal{D}. \tag{3}$$

We consider, for visual aids, the particular case.

Let us assume that in the Minkowsky space-time there exists a group of transformations which is locally isomorphic to the group $SO(p,q)$ ($p+q>6$, $p\geq 2$, $q\geq 4$). with generators $\mathcal{L}_{AB}$ (A,B = 0,1,2,3,5,6,7,..., $p+q$), where $\mathcal{L}_{ab} = L_{ab}$. Since

$$[\mathcal{L}_{AB}, \mathcal{L}_{CD}] = i(g_{AD}\mathcal{L}_{BC} + g_{BC}\mathcal{L}_{AD} - g_{AC}\mathcal{L}_{BD} - g_{BD}\mathcal{L}_{AC}),$$

where g_{AB} = (1,-1,-1,-1,-1,1,...), among the generators $\mathcal{L}_{AB}$ at least one generator Q (for example $\mathcal{L}_{67}$) which commute simultaneously with $L_{\mu\nu}$ and $L_{5\mu}$, which form algebra $SO(1,4)$, always exist. Let us find a form of such generator. From the condition $[Q, L_{\mu\nu}] = 0$ follows that

$$Q(x) = i\,\Phi(x^2)\, x_\nu \frac{\partial}{\partial x^\nu}, \tag{4}$$

where $\Phi(x^2)$ is arbitrary scalar function of $x^2 = x_0^2 - \vec{x}^2$. Then from the condition $[Q, L_{5\mu}] = 0$ we find, taking into account (2-4):

$$(1-x^2) g_{\mu\rho}\Phi(x^2) + 2x_\mu x_\rho[(1-x^2)\Phi'(x^2) + \Phi(x^2)] = 0, \tag{5}$$

$$\Phi'(z) = \frac{\partial \Phi(z)}{\partial z}; \quad \mu, \rho = 0,1,2,3.$$

From the equations (5) we obtain $(1-x^2)\Phi(x^2) = 0$, and, therefore the solution of the equations (5) is $\Phi = 0$*.

Thus, the generator $Q=0$, i.e. in the Minkowsky space-time there no exists an operator of the form (1) which commute simultaneously with $M_{\mu\nu}$ and $P_\mu + K_\mu$. Therefore, the group $SO(p,q)$ ($p+q>6$, $p\geq 2$, $q\geq 4$) cannot be realize as a group of coordinate transformations in the Minkowsky space-time.

The proof listed above is generelazed to arbitrary finite continuous group which contain the conformal subgroup. Indeed, any finite Lie group is locally isomorphic to some linear group [4]. And for linear groups as it isn't difficult to show at least one generator which does not belong to the subalgebra $SO(2,4)\sim SU(2,2)$ and commute with generators $L_{\mu\nu}$ and $L_{5\mu}$ of the subgroup $SO(1,4)\sim USp(2,2)$ (where $SO(1,4)\subset SO(2,4)$) always exists. But as we have seen such generator cannot be realized in the Minkowsky space-time.

Thus, the finite continuous group which contain the conformal group as a subgroup cannot be realized as the group of coordinate transformations in the Minkowsky space-time. Therefore, there no exists finite continuous group

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* We exclude singular solution $\Phi(x^2) = \delta(1-x^2)\,\varphi(x^2)$, where $\delta(z)$ is a delta-function, $\varphi(x^2)$ is arbitrary function.

of space-time symmetry higher than conformal *.

There exist at least one infinite continuous group, which contain the conformal group as a subgroup - it is the group of general coordinate transformations, recently considered in [5]. We would like to note, however, that the structure of general coordinate transformations group is essentially distinguished from the structure of Lie groups (for example, $SO(p,q)$, $SU(p,q)$) in the formal limit of infinite number of parameters ($p,q\to\infty$).

We see, thus, that finite group of symmetry can fix only small part of the dynamics, maximally two- and three-point functions (conformal group). Therefore, in order to fix the more part of the dynamics we must consider an infinite groups.

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* For space with a metric which contain n plus and m minus the maximal group of space symmetry compatible with Lorentz invariance is the conformal group of this space, i.e. the group $SO(n+1,m+1)$.

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