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IN THE PERTURBATION THEORY

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Abstract

Within the framework of the perturbation theory the theorem pointed out by Bjorken, Johnson and Low which connects T-products and equal-time commutators is considered. In the cases when the matrix elements of T-product and that of equal-time commutator are finite this theorem is always true. However, the equal-time commutators do not coincide, generally speaking, with "naive" ones and depend upon the type of interaction.

In the paper it is shown on simple examples that the equal-time commutation relations of field operators remain canonical. The other equal-time commutators may be, in principle, calculated with the help of canonical commutation relations if taking into account the singular character of the products of the field operators at coinciding points.

It is shown also that in some cases the using of the Feynman rules for the calculation of T-products leads to a wrong result, that is connected with the inapplicability of Wick theorem for formally divergent diagrams.

1. Introduction

Bjorken 1 and Johnson and Low 2 showed that the asymptotics of the matrix element of the chronological product of two operators

It follows from here that, firstly, value T in the considered limit decreases and, secondly, that

lim P.T = Sdx e'PX S(x.) (C[A(x), B(o)]|a) (3)
The theorem of Bjorken, Johnson and Low (BJL) gives,
at first sight, the convenient way of finding asymptotics of amplitudes, since a priori it may be
expected that the equal-time commutators do not
depend upon the type of strong interactions.

In some works 3-5 relation (3) was checked in the limits of the perturbation theory in various field models of strong interaction. It was shown 3, in particular, that for the validity of BJL theorem it is necessary that the matrix elements of T-product

as well as that of commutator were finite. It is easy to understand that this criterion is also sufficient one.

However, in the case of finiteness of all quantities it is occured that the equal-time commutator in (3) does not coincide, generally speaking, with "naive" one, that is, with one calculated by formal application of canonical commutation relations. Non-coincidence of asymptotics of T-product with the matrix element of "naive" commutator was considered in works 4,5 as the failure of BJL theorem, though, strictly speaking, relation (3) does not fail and the problem lies in the distinction of the commutator from "naive" one. The reason of this distinction lies in the singular character of operators A and B which are the products of field operators in coinciding points. As the classical example of such a kind serves the commutator of the components of vector current of fermions, that does not coincide with "naive" one as was shown by Schwinger/6/.

In the work of Jackiw and Preparata 4 the asymptotics of fermion propagator is considered which is determined by vacuum matrix element of anticommutator S(x) = S

this propagator in the second order in a coupling constant is finite in Landau gauge. However, its asymptotics appears to be different from that predicted by relation (3) and canonical commutators. In this case it is impossible to explain the arising contradiction by the singular character of operators entering commutator. Moreover, the failure of canonical commutation relations suggested by authors would question the logical consistency of the field theory. In addition it would deprive us of the possibility to calculate directly other equal-time commutators.

In the next section it will be shown that the reason of the mentioned contradiction is the incorrect calculation of T-product. The use of the Feynman rules for the determination of Green function leads to incorrect result, that is connected with the inapplicability of Wick theorem in the case of formally divergent diagrams (even if the result is finite).

Let us note that inapplicability of the Feynman rules for a calculation of some T-product results only in distinction of renormalization constants and, therefore, does not change the value of amplitude of physical processes.

In the third section the examples of commutators which do not coincide with "naive" ones are considered. It is shown that after taking into account the singular character of the products of field operators the using of canonical commutation relations leads to the results agreeing with theorem (3).

2. The asymptotics of mass operator of fermion and the canonical commutation relations

Let us consider the asymptotics of fermion Green function

 $G(p)=-i\int dx e^{ipx}\langle 0|T\Psi(x)\overline{\Psi(0)}|0\rangle$ (4) in the second order of the perturbation theory in the model of strong interactions mediated by neutral vector meson with mass μ . The fermion field is supposed to be zero-mass throughout the paper.

Since in the second order of the perturbation theory Green function G(p) is finite in Landau gauge, then, according to the criterion mentioned, one may expect that because of BJL theorem

$$\lim_{p\to\infty} \hat{p} G(p) = \chi_0 \int dx e^{ip\chi} \delta(x_0) \langle 0| \{ \Psi(x), \overline{\Psi}(0) \} | 0 \rangle = 1$$
In the last equation canonical commutation rela-

tion is used

$$\delta(x,0) \{ \Psi(x), \overline{\Psi}(0) \} = \gamma \delta(x)$$
 (6)

On the other hand

$$G(p) = \frac{1}{\beta - \Sigma(p)}$$
, (7)

where in the second order in the coupling constant

$$\sum (p) = -ig^2 \int dx e^{ipx} \langle 0|T J(x)J(0)|0\rangle$$
 (8)

In eq.(8) fermion current J(x) = J(x)Y(x) where field operators H(x), Y(x) are the operators of free fields. By means of Wick theorem eq.(8) is reduced to the following one

$$\sum (p) = -\frac{ig^2}{(2\pi)^n} \int dk (g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}) \frac{1}{k^2 - m^2} \gamma_{\mu} \frac{1}{\beta - \hat{k}} \gamma_{\nu}, \quad (9)$$

that meets conventional Feynman diagram for the mass operator of fermion.

Calculating after Feynman parametrization the integral over & and changing the variable in integral over Feynman parameter we obtain after some transformations

$$\sum (p) = \frac{g^2}{\pi} \hat{p} \left\{ \int dx^2 \frac{x^2 G'(x^2)}{p^2 - x^2 + i\epsilon} + \int dx^2 G'(x^2) \right\} =$$

$$= \frac{g^2}{\pi} \hat{p} \left\{ \int dx^2 \frac{x^2 G'(x^2)}{p^2 - x^2 + i\epsilon} - \frac{3}{32\pi} \right\} \tag{10}$$

$$6'(x^2) = -\frac{3}{32\bar{1}} \frac{\mu^2}{x^4} \left(1 - \frac{2}{3} \frac{\mu^2}{x^6}\right) \theta(x^2 - \mu^2) - \frac{1}{32\bar{1}\mu^2} \left[\theta(x^2) - \theta(x^2 - \mu^2)\right]$$

At p > 0 the first term in this expression for decreases so that

$$\lim_{p \to \infty} \sum (p) = -\frac{39^2}{32\pi^2} \hat{p}$$
 (11)

Thus, for Green's function we obtain 4/

lim $\hat{p}G(p) = \left(1 - \frac{39^2}{327^2}\right)$ in contradiction to eq.(5). Since as stated in the introduction we assert that the asymptotics of T-product is determined by equal-time anticommutator of fermi-field operators we should assume that $\frac{4}{4}$

$$\delta(x_{\bullet})\langle 0|\{\Psi(x),\overline{\Psi}(0)\}|0\rangle = \gamma_{\bullet}\left(1 - \frac{39^{2}}{32\overline{1}^{2}}\right)\delta'(x) \qquad (13)$$
in contradiction to canonical relation (6).

The reason of this contradiction lies in incorrect calculation of $\sum (p)$. Indeed, mass operator $\sum (p)$ which is expressed through T-product of operators I and is finite in our case must decrease* at $p \to \infty$. Meanwhile, it is seen from eq.(11) that is not the case.

Thus, the failure of BJL theorem had occured already in calculation of $\sum (p)$. Moreover, the calculation of $\sum (p)$ is ambiguous. Really, though $\sum (p)$ is a finite quantity in Landau gauge the integral over momentum in eq.(9) formally diverges linearly. Therefore, making, for example, the shift of integration variable $p \rightarrow p + k$ we obtain the equation

$$\sum (p) = \frac{9^2}{\pi} \hat{p} \int dx^2 \frac{x^2 6(x^2)}{p^2 - x^2 + i\epsilon}$$
 (14)

differing from (10) by polinome. Now $\sum (\infty) = 0$ and there is no contradictions to the canonical anti-commutator (6).

Let us show that the correct equation for $\sum (p)$ coincides with (14). Really, according to the definition of T-product used in the proof of BJL theorem, it is necessary to calculate, firstly, the products of operators and then to take into account θ -functions. In our case it gives

$$\sum (p) = -ig^{2} \int dx e^{ipx} \left\{ \theta(x_{0}) \chi_{\mu} i S(x_{0}) \chi_{\nu} i \Delta_{\mu\nu}^{(\mu)}(x) + \theta(-x_{0}) \chi_{\nu} i S(x_{0}) \chi_{\mu} i \Delta_{\mu\nu}^{(\nu)}(x) \right\} = \frac{1}{2\pi i} \int_{\tau - i\varepsilon}^{\tau - i\varepsilon} \sum (p, \tau)$$
where

^{*} Though matrix element $\delta(x.)$ $\langle 0|\{J(x), J(0)\}|0\rangle$ diverges logarithmically in the considered case the mentioned property of T-product retains.

and we used Fourier-representation of step-function

 $\Theta(x_o) = \Theta(nx) = \frac{1}{2\pi i} \int_{0}^{\infty} d\tau \frac{e^{i\tau nx}}{\tau - i\epsilon}$ (17)

(N_{μ} is a unit time-like vector, $N_{\mu} = (1,0,00)$, $S^{(\frac{1}{2})}(x)$, $S^{(\frac{1}{2})}(x)$ is positive- and negative-frequency parts of invariant commutation functions of fermion and boson fields.

Integral (16) converges and is defined unambiguously. As a matter of fact, it coincides with an imaginary part of a mass operator so that (15) gives the spectral representation for $\sum (p)$. After calculation of $\sum (p,T)$ changing a variable in integral over parameter \mathcal{T} by simple way we come to the result coinciding with (14).

The above consideration shows that the application of Wick theorem to a formally divergent diagram leads to the incorrect result. The point is that the equations of the type of $\theta(x_0) + \theta(-x_0) = 1$, $\theta(x_0) + \theta(-x_0) = 0$ which are used when proving this theorem occur to be invalid if they are integrated with function singular at $x_0 = 0$.

3. "Non-naive" commutators and canonical commutation relations

In this section the examples of "non-naive" commutators are considered: $\delta(x_*)[\Psi(x), j_{\mu}(o)]$ and $\delta(x_*)[J(x), j_{\mu}(o)]$ in the model of strong interactions used above. It is shown that under accurate

treatment of the singular products of the field operators these commutators may be, in principle, calculated with the help of canonical commutation relations.

commutator $S(X_0)[\Psi(X), J_{\mu}(0)]$ determines the asymptotics of amplitude

$$F_{\mu}(p) = -i \int dx e^{ipx} \langle 0|T\Psi(x)j_{\mu}(0)|p'=0 \rangle$$
 (18)

at $p_* \rightarrow \infty$. To simplify the calculation we set the four-momentum of initial fermion to be equal to zero.

We find the commutators of interest calculating the quantity $F_{\mu}(p)$. As it was noted the using of the Feynman rules for the calculation of T-products leads, generally speaking, to the incorrect result. Therefore, we shall operate in the same way as in the calculation of $\sum (p)$ (see eq.(15) and (16)), that is, represent $F_{\mu}(p)$ as

$$F_{\mu}(p) = -\frac{1}{2\pi} \int_{-12}^{\infty} \frac{d\tau}{\tau - i\epsilon} \left[F_{\mu}(p + n\tau) + F_{\mu}(p - n\tau) \right] (19)$$

where

$$F_{\mu}^{\dagger}(p) = \int dx e^{ipx} \langle 0| \Psi(x) j_{\mu}(0) | p' = 0 \rangle$$
 (20)
 $F_{\mu}^{\dagger}(p) = \int dx e^{ipx} \langle 0| j_{\mu}(0) \Psi(x) | p' = 0 \rangle$

Quantities $f_{\mu}^{\dagger}(p)$ determining the imaginary Part are calculated inserting the complete set of intermediate states between operators Y and f_{μ} . For example, in the calculation of value f_{μ}^{\dagger} up to the second order in g the states with one fermion and with fermion and boson are essential. The contribution of two-partcile intermediate state is calculated unambiguously. One-particle state gives

 $2T\delta(p^2)\theta(p)\sum_{spin}\langle o|\Psi(o)|p\rangle\langle p|j_{\mu}(o)|p=0\rangle$, (21) Here matrix elements are described by the diagrams with loops and, therefore, demand definition. In particular, quantity $\langle o|\Psi(o)|p\rangle$ is determined by the mass operator of fermion and, as it follows from the results of previous section, is equal to

$$\langle 0|\Psi(0)|p\rangle = Z_{u}^{\frac{1}{2}} = (1 + \frac{3g^{2}}{32T^{2}})^{\frac{1}{2}}$$
 (22)

Matrix element $\langle p | j_{\mu}(o) | p'=0 \rangle$ may be found from the vector current conservation leading to non-renormalization of this matrix element at (p-p')=0

$$\langle p|j_{\mu}(0)|p'=0\rangle = \overline{U}(p)\gamma_{\mu}U(p'=0)$$
 (23)
As a result, we obtain the next expression for $F_{\mu}(p)$

$$\hat{p}F_{\mu}(p) = Z^{2}_{\chi} u + \frac{4g^{2}}{3\pi} (p^{2}_{\mu} - \hat{p}p_{\mu}) \int dx^{2} \frac{\delta(x^{2})}{p^{2} - x^{2}_{+}i\epsilon} u, \quad (24)$$
Where $\delta(x^{2})$ is the

where $6(x^2)$ is the same spectral function as in eq.(10).

Note, that the direct calculation of $F_{\mu}(p)$ by the Feynman rules leads to the equation which differs from (24) by substitution unity for Z.

Commutator $\delta(x.)\langle 0|[\Psi(x),j_{\rho}(0)]|p=0\rangle$ may be found with the help of relation

$$\hat{p} F_{\mu}(p) = \lambda_{0} \int dx e^{ipx} \delta(x_{0}) \left\langle 0 \left[\left[\Psi(x) \right] \right] p'=0 \right\rangle - ig \int dx e^{ipx} \left\langle 0 \right] T J(x) \int_{\mu} (0) \left[p'=0 \right\rangle - ig \int dx e^{ipx} \left\langle 0 \right] T J(x) \int_{\mu} (0) \left[p'=0 \right\rangle ,$$
where due to BIT therefore the second series of the second series are the second series of the second second series of the second second series of the second second series of the second secon

where due to BJL theorem the second term in the right-hand side decreases at $p \to \infty$. Comparing eqs.(25) and (24) in this limit we come to the conclusion that

$$\int dx \, e^{ipx} \delta(x_0) \langle 0| [\Psi(x), j_{\mu}(0)] | p'=0 \rangle =$$

$$= \lambda_0 \left[\frac{1}{2} \gamma_{\mu} + (\gamma_{\mu} - n_{\mu} \hat{n}) \frac{49^2}{3\pi} \int dx^2 \, \delta(x^2) \right] u \qquad (26)$$
From (24-26) we obtain also

$$T_{\mu}(p) = -i \int dx e^{ipx} \langle 0|TJ(x)j_{\mu}(0)|p'=0 \rangle =$$

$$= \frac{49}{3\pi} \left[(p^{2}\gamma_{\mu} - \hat{p}p_{\mu}) \int dx^{2} \frac{\delta(x^{2})}{p^{2} - x^{2} + i\pi} - (\gamma_{\mu} - n_{\mu}\hat{n}) \int dx^{2} \delta(x^{2}) \right] u$$
(27)

^{*} We do not take into account the contribution of diagram of vacuum polarization. It is absent if one considers isovector current in the theory in which fermions with isospin 1/2 interact with isoscalar vector meson.

The calculation of Tp (p) by the Feynman rules does not give a non-covariant second term resulting in the incorrect behaviour of Tp(p) at p -> 00.

With the help of BJL theorem we obtain from

$$\int dx e^{ipx} \int (x.) \langle 0| [7(x), j.(0)] | p'=0 \rangle = -\frac{9}{8\pi^2} \vec{p} \vec{y} u \qquad (28)$$

$$\int dx e^{ipx} S(x.) \langle 0| [J(x), j(0)] | p'=0 \rangle = \frac{9}{8\pi^2} \vec{p} \gamma_0 u$$
 (29)

Now let us show how to obtain commutators (26,28,29) by means of canonical commutation relations. Their formal application leads to value Z'2 Y. Y, U for commutator (26) and to zero result in the case of commutators (28,29). Thus, only commutator $\delta(x_0)[\Psi(x), J_0(0)]$ coincides with "naive" one.

Such calculation is incorrect, however, since we deal with the singular products of the field operators in coinciding points. Commutator (28) may serve as an example to show how this circumstance is displayed. With the help of canonical commutation relations (6) we have

$$\frac{\delta(x_0)\langle 0| L^{3}(x), j_{\bullet}(0)]| p'=0\rangle =}{= \delta'(x) \langle 0| \hat{R}(x) \Psi(0)| p'=0\rangle = \delta'(x) I(x) U}$$
(30)

If in matrix element $\langle 0|\hat{A}(0,\vec{x})|\Psi(0)|p'=0\rangle$ put X = 0 taking into account factor $\delta^3(\vec{x})$ one may obtain matrix element of fermion source $J(0) = \tilde{A}(0)\Psi(0)$

between vacuum and one-particle state which is equal to zero. But, the operators product $\hat{A}(x)\Psi(0)$ has singular character at $x \rightarrow 0$ and matrix element of it is not integrable with & function. Indeed, in the first order of the perturbation theory at $X_* = 0$ but $X \neq 0$

$$I(x) = \frac{1}{(2\pi)^{4}} \int dk e^{-ikx} I(k) = \frac{9}{(2\pi)^{4}} \int dk e^{ikx} I(k) \frac{(9\mu\nu - \frac{k\mu k\nu}{k^{2}})}{k^{2} - \mu^{2}} (31)$$

In the limit $\vec{X} \rightarrow 0$ $I(x) \sim \vec{y} \vec{x} / \vec{x}^2$. To obtain an explicit form of the singularity of the commutator it is convenient to pass in (33) to the momentum representation. If the integration over momentum corresponding to function $\delta'(X)$ is postponed to the end of calculation we obtain according to (28)

$$\int dx e^{ipx} \delta(x,0) \langle 0| [\Im(x),j,(0)] | p'=0 \rangle =$$

$$= \frac{i}{(2\pi)^4} \int dq \, I(-q,p-q) u = -\frac{q}{8\pi^2} \vec{p} \vec{q} u \qquad (32)$$

Integral over 9 is calculated in the following order: at first, the integration over 9, is carried out, then that in spatial component. Such succession of calculations corresponds to taking firstly X, = 0 .

The used method is similar to the Schwinger procedure of definition of the operator products, but it does not demand the introduction of additional spatial-like vector & . It is clear that tional spatial-like vector & . It is clear that such method gives the possibility to calculate /7/ such method gives the possibility to calculate current density without introduction of & .

As to the commutator $\delta(x.) \langle o|[\Psi(x), j.(o)]|p=o\rangle$ in its calculation, in contrast to the above case, there occur no additional singularities and the commutator remains "naive".

In the calculation of commutators $\Psi(x)$ and $\Upsilon(x)$ with spatial components of current χ_m it is necessary to take into account explicit dependence of χ_m upon the operator of a vector field χ_m . As it is shown in work χ_m the corresponding additions to the current density and hamiltonian are expressed in interaction representation through repeated commutators of the components of the current density of non-interacting fields. All arising equal-time commutators may be calculated by usual method, but the using of χ_m functions of the spatial variables must be postponed to the end of calculations, as it was done earlier. The rather tedious transformations connected with the necessi-

ty of the definition of arising ambiguities lead to the result coinciding with (26) and (29).

Note, that in the case of time component of the current density which is a generator of a gauge group the dependence upon a vector field is absent.

Let us take notice of the analogy between the considered commutator $\delta(x_0) \langle 0|[f(x), f(0)]| \psi_0' 0 \rangle$ and the commutator of components of the vector current density $\delta(x_0) \langle 0|[f_m(x), f(0)]| 0 \rangle$. Their non-zero value is due to the Schwinger terms which disappear after integrating over spatial variables. The presence of thexterms leads to the appearance of "non-naive" additions in commutators $\delta(x_0) \langle 0|[\psi(x), f_m(0)]|\psi_0' \rangle$ and $\delta(x_0) \langle 0|[f_m(x), f_n(0)]|0 \rangle$

4. Conclusion

The results of the calculations carried out in this article show that BJL theorem in the perturbation theory does not fail if neither T-product nor corresponding commutator have divergences.

The considered examples show that the Feynman rules even for diagrams with well-defined, that is non-containing loops, imaginary part give for T-product the result which is true only up to polinome in momentum. It is connected with the failure of Wick's theorem in the case of formally

divergent diagrams. In these cases the correct result decreasing in the limit of β , $\rightarrow \infty$ is obtained automatically if one acts according to the definition of the chronological product.

When an imaginary part contains loops the question arises about the determination of the amplitudes due to these loops. In the example amplitudes due to these loops. In the example considered in the second section of this article such amplitude is $\sum (p)$ coinciding with the matrix element of T-product of fermion sources and thus defined unambiguously by the condition of decreasing at $p \to \infty$.

But such situation is not general. For example, while considering Green function of a vector field in the second order, there arises the necessity of definition of the polarization operator which as it is known does not coincide with the vacuum matrix element of T-product of currents. Analogously matrix element $\langle p|j_{\mu}(o)|p'\rangle$ which is necessary for calculation of f_{μ} is not represented as chronological product

-ig Saxeipx (olT J(x) jn (o) /p'>

The above consideration allows, in particular, to eliminate the stated in work contradiction between BJL theorem and canonical commutation relations. The equal-time commutators of more complex operators may be calculated, in principle,

with the help of these relations. In these calculation it is necessary to take into account the singular character of the products of field operators and the explicit dependence of current upon vector field caused by this circumstance. Such equal-time commutators occur to be dependent upon the interaction and their calculation is, generally speaking, a problem not less complicated as that of finding of asymptotics of corresponding T-products.

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